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Editorial Introduction

This volume contains the Proceedings of the International Workshop on Operator Theory and Applications (IWOTA) which was held at the Seoul National University, Seoul, Korea, July 31–August 3, 2006. This was the seventeenth IWOTA; in fact, the workshop was held biannually since 1981, and annually in the recent years (starting 2002) rotating among twelve countries on four continents.

Previous IWOTA meetings were held at:

Santa Monica, CA, USA (1981) J.W. Helton, Chair

Rehovot, Israel (1983) – OT 12 H. Dym and I. Gohberg, Co-chairs

Amsterdam, Netherlands (1985) – OT 19 M.A. Kaashoek, Chair

Mesa, AZ, USA (1987) – OT 35 J.W. Helton and L. Rodman, Co-chairs

Rotterdam, Netherlands (1989) – OT 50 H. Bart, Chair

Sapporo, Japan (1991) – OT 59 T. Ando, Chair

Vienna, Austria (1993) – OT 80 H. Langer, Chair

Regensburg, Germany (1995) – OT 102 and 103 R. Mennicken, Chair

Bloomington, IN, USA (1996) – OT 115 H. Bercovici and C. Foias Co-chairs

Groningen, Netherlands, (1998) – OT 124 A. Dijksma, Chair

Bordeaux, France (2000) – OT 129 N. Nikolsky, Chair

Faro, Portugal (2000) – OT 142 A.F. Dos Santos and N. Manojlovic, Co-chairs

Blacksburg, VA, USA (2002) – OT 149 J. Ball, Chair

Cagliari, Italy (2003) – OT 160 S. Seatzy and van der Mee, Co-chairs

Newcastle, UK (2004) – OT 171 M.A. Dritshel and N. Young, Co-chair

Storrs, CT, USA (2005) – OT 179 V. Olshevsky, Chair

Seoul, Korea (2006) – OT in print Woo Young Lee, Chair

Potchefstroom, South Africa (2007) – OT in preparation, K. Grobler and G. Groenewald, Co-chairs

Williamsburg, VA, USA, (2008) – OT planned, L. Rodman, Chair

The purpose of IWOTA 2006 was to bring together mathematicians and engineers interested in operator theory and its applications. Adhering to the tradition of the recent IWOTA meetings, IWOTA 2006 was focused on a few special themes, without loss of the general IWOTA mission. Our special interest areas were

Hilbert/Krein space operator theory;

Complex function theory related to Hilbert space operators;

Systems theory related to Hilbert space operators.

This volume contains 16 contributions, which reflect the recent development in operator theory and applications.

The organizers gratefully acknowledge the support of the following institutions:

KRF (Korea Research Foundation);

Department of Mathematics, Seoul National University;

Research Institute of Mathematics, Seoul National University.

Tsuyoshi Ando, Raúl Curto
Il Bong Jung, Woo Young Lee
(Editors)

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A Connection between Szegö and Nehari Sequences in the Matrix-valued Case

Daniel Alpay and Israel Gohberg

Abstract. One can associate to a rational function which is moreover strictly positive on the unit circle two sequences of numbers in the open unit disk, called the Szegö sequence and the Nehari sequence. In the scalar case, they coincide up to multiplication by -1. We study the corresponding result in the matrix-valued case.

Mathematics Subject Classification (2000). Primary: 34A55, 49N45, 70G30; Secondary: 93B15, 47B35.

Keywords. Inverse problems, scattering matrix, Schur parameters, state space method, extension problems.

1. Introduction

Let $w(z)$ be a scalar rational function strictly positive on the unit circle. One can associate to it an infinite sequence of numbers in the open unit disk, called in [1] a Szegö sequence. This sequence characterizes in a unique way $w(z)$ provided some normalization is chosen; we will take

$$\frac{1}{2\pi}\int_0^{2\pi} w(e^{it})dt = 1.$$

Similarly, a rational function $s(z)$ unitary on the unit circle and which admits a spectral factorization can be characterized in a unique way (after an appropriate normalization) by another infinite sequence of numbers in the open unit disk, called in [1] a Nehari sequence. Szegö sequences and Nehari sequences are closely related. More precisely, let $s_+(z)$ be a rational function analytic and invertible in

Daniel Alpay wishes to thank the Earl Katz family for endowing the chair which supported his research.

an open neighborhood of the closed unit disk, and such that $s_+(0) = 1$. We proved in [4] that the Szegö sequence associated to $w(z) = c/|s_+(z)|^2$ with

$$c = \frac{2\pi}{\int_0^{2\pi} |1/s_+(e^{it})|^2 dt},$$

coincides with the opposite of the Nehari sequence associated to $zs(z)$, where

$$s(z) = s_+(z)/(\overline{s_+(1/\overline{z})}).$$

The purpose of the present paper is to prove a similar result in the matrix-valued case. Before presenting the matrix-valued case, we review the main definitions and the main result in the scalar case. Let thus $w(z)$ be a rational function without poles on the unit circle and strictly positive there. Then

$$w(z) = \sum_{\mathbb{Z}} r_\ell z^\ell, \quad \text{for} \quad |z| = 1, \tag{1.1}$$

where $\sum_{\mathbb{Z}} |r_\ell| < \infty$. Since $w(z) > 0$ on the unit circle, the Toeplitz matrices

$$T_n = \begin{pmatrix} r_0 & r_{-1} & \cdots & r_{-n} \\ r_1 & r_0 & \cdots & \\ \vdots & & & \vdots \\ r_n & & \cdots & r_0 \end{pmatrix}, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

are strictly positive. We set $T_n^{-1} = (\gamma_{\ell j}^{(n)})_{\ell,j=0,\ldots,n}$. The Szegö sequence is defined by

$$\alpha_n = \gamma_{n0}^{(n)} (\gamma_{00}^{(n)})^{-1} \quad n = 1, 2, \ldots.$$

It is also called the sequence of Verblunsky coefficients or the sequence of reflection coefficients associated to $w(z)$.

Let now $s(z)$ be a rational function unitary on the unit circle, and assume that $s(z)$ admits a spectral factorization: $s(z) = s_+(z)/(\overline{s_+(1/\overline{z})})$, where $s_+(z)$ and its inverse are analytic in an open neighborhood of the closed unit disk. We set

$$s(z) = \sum_{\mathbb{Z}} \gamma_n z^n, \quad |z| = 1. \tag{1.3}$$

Then, the Toeplitz operator with symbol $s(z)$ is invertible (see [9, Theorem 4.1, p. 588]), and hence the Hankel operators

$$\Gamma_n = \begin{pmatrix} \gamma_{-n} & \gamma_{-n-1} & \cdots \\ \gamma_{-n-1} & \gamma_{-n-2} & \cdots \\ \vdots & \vdots & \\ \vdots & \vdots & \end{pmatrix}, \quad \ell_2 \to \ell_2, \quad n = 0, 1, 2, \ldots \tag{1.4}$$

are strictly contractive. Let

$$a^{(n)} = \begin{pmatrix} a_{n0} \\ a_{n1} \\ \vdots \end{pmatrix} = (I_{\ell^2} - \Gamma_n \Gamma_n^*)^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \in \ell_2.$$

The sequence of numbers

$$\nu_n = -(\overline{\gamma_{-n}} a_{n+1,0} + \overline{\gamma_{-n-1}} a_{n+1,1} + \cdots), \quad n = 0, 1, \dots$$

is called the Nehari sequence associated to $s(z)$. All the ν_n are in the open unit disk. This sequence was first defined in [8].

The next theorem is given, in an equivalent form, in [4]; see Theorem 3.1 in that paper.

Theorem 1.1. *Let $s_+(z)$ be a rational function analytic and invertible in the closed unit disk, and such that $s_+(0) = 1$. Define functions $s(z)$ and $w(z)$ by:*

$$s(z) = s_+(z)/\overline{s_+(1/\overline{z})}, \quad \text{and} \quad w(z) = \frac{c}{|s_+(z)|^2}, \quad \text{with} \quad c = \frac{2\pi}{\int_0^{2\pi} |1/s_+(e^{it})|^2 dt},$$

and let $(\nu_n)_{n\geq 0}$ and $(\alpha_n)_{n\geq 1}$ be the Nehari sequence and the Szegö sequence associated to $zs(z)$ and $w(z)$ respectively. Then,

$$\alpha_n = -\nu_{n-1}, \quad n = 1, 2, \dots.$$

We now present the matrix version of this theorem. We first introduce the matrix version of the Szegö and Nehari sequences. Now we have sequences of pairs of matrices, rather than sequences of numbers. To define Szegö sequences in the matrix-valued case, consider a $\mathbb{C}^{p\times p}$-valued rational function which takes strictly positive values on the unit circle, and with representation (1.1), where now the r_ℓ are in $\mathbb{C}^{p\times p}$. The matrices T_n defined by (1.2) are now block Toeplitz matrices; as in the scalar case they are strictly positive. We set $T_n^{-1} = (\gamma_{\ell j}^{(n)})_{\ell,j=1,\dots,n}$ to be the block decomposition into $\mathbb{C}^{p\times p}$ blocks of the inverse of T_n. The sequence (α_n, β_n), $n = 1, 2, \dots$ defined by

$$\alpha_n = \gamma_{n0}^{(n)} (\gamma_{00}^{(n)})^{-1} \quad \text{and} \quad \beta_n = \gamma_{0n}^{(n)} (\gamma_{nn}^{(n)})^{-1}, \quad n = 1, 2, \dots \tag{1.5}$$

is called the Szegö sequence associated to $w(z)$. In the scalar case we have $\alpha_n = \overline{\beta_n}$.

To define Nehari sequences in the matrix-valued case we start with a $\mathbb{C}^{p\times p}$-valued rational function $s(z)$, unitary on the unit circle, and which admits a spectral factorization, and write $s(z)$ as (1.3), where now the γ_j are matrices in $\mathbb{C}^{p\times p}$. The Hankel operators Γ_n now act from $\ell_2^{p\times p}$ into itself, and are strictly contractive. Consider the solutions $a^{(n)}, b^{(n)}, c^{(n)}, d^{(n)} \in \ell_2^{p\times p}$ of the equations

$$\begin{pmatrix} I_{\ell_2^{p\times p}} & -\Gamma_n \\ -\Gamma_n^* & I_{\ell_2^{p\times p}} \end{pmatrix} \begin{pmatrix} a^{(n)} \\ c^{(n)} \end{pmatrix} = \begin{pmatrix} e \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_{\ell_2^{p\times p}} & -\Gamma_n \\ -\Gamma_n^* & I_{\ell_2^{p\times p}} \end{pmatrix} \begin{pmatrix} b^{(n)} \\ d^{(n)} \end{pmatrix} = \begin{pmatrix} 0 \\ e \end{pmatrix}, \tag{1.6}$$

where

$$e = \begin{pmatrix} I_p \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

We set

$$a^{(n)} = \begin{pmatrix} a_{n0} \\ a_{n1} \\ \vdots \end{pmatrix}, \quad b^{(n)} = \begin{pmatrix} b_{n0} \\ b_{n1} \\ \vdots \end{pmatrix}, \tag{1.7}$$

and similarly for $c^{(n)}$ and $d^{(n)}$. The sequence (α_n, β_n), $n = 1, 2, \ldots$ where $\alpha_n = \rho_{n-1}$ and $\beta_n = \nu_{n-1}$ $n = 1, 2, \ldots$ and where ρ_n and ν_n are given by

$$\begin{aligned} \nu_n &= -(\gamma_{-n}^* a_{n+1,0} + \gamma_{-n-1}^* a_{n+1,1} + \cdots) \\ \rho_n &= -(\gamma_{-n} d_{n+1,0} + \gamma_{-n-1} d_{n+1,1} + \cdots) \end{aligned}$$

is called the Nehari sequence associated to $s(z)$. In the scalar case we have $\nu_n = \overline{\rho_n}$. Such sequences were introduced in [8]. Before stating the main result of this paper we need some more notation. We denote by $\mathcal{W}$ the Wiener algebra of functions of the form

$$f(z) = \sum_{\mathbb{Z}} z^n f_n \qquad \text{with} \qquad \sum_{\mathbb{Z}} |f_n| < \infty$$

By $\mathcal{W}_+$ (resp. $\mathcal{W}_-$) we denote the subalgebra of $\mathcal{W}$ consisting of the functions $f(z)$ for which $f_n = 0$ for $n < 0$ (resp. $n > 0$). The symbol $\mathcal{W}^{p\times p}$ denotes the algebra of $p \times p$ matrices with entries in $\mathcal{W}$; the subalgebras $\mathcal{W}_+^{p\times p}$ and $\mathcal{W}_-^{p\times p}$ are defined similarly. We already used earlier in the paper (see (1.1) and (1.3)) the fact that a rational function with no pole on the unit circle belongs to $\mathcal{W}$.

Theorem 1.2. *Let $w(z)$ be a $\mathbb{C}^{p\times p}$-valued rational function strictly positive on the unit circle, let*

$$w(z) = \sum_{\mathbb{Z}} r_\ell z^\ell, \quad |z| = 1,$$

and assume $r_0 = I_p$. Let $T_n = (r_{\ell-j})_{\ell,j=0,\ldots,n}$ and $T_n^{-1} = (\gamma_{\ell j}^{(n)})_{\ell,j=0,\ldots,n}$, where $\gamma_{\ell,j}^{(n)} \in \mathbb{C}^{p\times p}$, and let

$$\alpha_n = \gamma_{n0}^{(n)} (\gamma_{00}^{(n)})^{-1} \quad \text{and} \quad \beta_n = \gamma_{0n}^{(n)} (\gamma_{nn}^{(n)})^{-1}$$

be the Szegö sequence associated to $w(z)$. Set

$$w(1/z) = s_-(z)^{-1} \delta_1^{-1} s_-(z)^{-*} = s_+(z) \delta_2^{-1} s_+(z)^*$$

where δ_1 and δ_2 are strictly positive $p \times p$ matrices, and where $s_-(z)$ and its inverse are in $\mathcal{W}_-^{p\times p}$, $s_+(z)$ and its inverse are in $\mathcal{W}_+^{p\times p}$ and $s_-(\infty) = s_+(0) = I_p$. Define

$$z\sigma(z) = z\sqrt{\delta_1} s_-(z) s_+(z) \sqrt{\delta_2}^{-1} = \sum_{\mathbb{Z}} \gamma_k z^k.$$

Let

$$\nu_n = -(\gamma_{-n}^* a_{n+1,0} + \gamma_{-n-1}^* a_{n+1,1} + \cdots)$$
$$\rho_n = -(\gamma_{-n} d_{n+1,0} + \gamma_{-n-1} d_{n+1,1} + \cdots)$$

be the Nehari sequence associated to $z\sigma(z)$. Assume that

$$\lim_{n\to\infty} \gamma_{nn}^{(n)} = \lim_{n\to\infty} \gamma_{00}^{(n)} \quad \text{and} \quad a_{0,0} = d_{0,0}. \tag{1.8}$$

Then, the Szegö sequence and the Nehari sequence are related by:

$$\alpha_n = -a_{00}^{1/2} \rho_{n-1} a_{00}^{-1/2} \quad \text{and} \quad \beta_n = -a_{00}^{1/2} \nu_{n-1} a_{00}^{-1/2}, \quad n = 1, 2, \ldots.$$

The proof is based on the theory of matrix-valued first-order systems, which we developed in [1] and [2]; see [5] and [6] for the scalar case. In the proof, recursion (2.13) plays a central role. We proved this recursion in the rational case in [4], and this is why we consider rational functions.

The paper consists of three sections besides the introduction. In the second section we review the part of the theory of matrix-valued first-order systems needed in the proof of Theorem 1.2. The proof of Theorem 1.2 itself is in the third section. In the last section we present an example where conditions (1.8) are in force.

2. First-order discrete systems

We recall in this section the results of the theory of first-order discrete systems needed in the proof of Theorem 1.2. We first define *admissible sequences*. Consider a sequence $\Delta = (\Delta_n)$, $n = 0, 1, \ldots$, of strictly positive block-diagonal matrices:

$$\Delta_n = \begin{pmatrix} \delta_{1n} & 0 \\ 0 & \delta_{2n} \end{pmatrix},$$

where δ_{1n} and δ_{2n} belong to $\mathbb{C}^{p\times p}$ and are strictly positive. Then (see [1, Definition 1.1]) the sequence of pairs (α_n, β_n) of $\mathbb{C}^{p\times p}$ matrices ($n = 1, 2, \ldots$) is said to be Δ-admissible if

$$\begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p \end{pmatrix} J\Delta_n \begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p \end{pmatrix}^* = J\Delta_{n-1}, \quad n = 1, 2, \ldots \tag{2.1}$$

where

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}.$$

The sequence is said to be normalized if $\Delta_0 = I_{2p}$.

In the scalar case one has necessarily $\alpha_n = \overline{\beta_n}$ and one can chose $\Delta_0 = I_2$ and

$$\Delta_n = \frac{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\prod_1^n (1 - |\alpha_j|^2)}, \quad n = 1, 2, \ldots. \tag{2.2}$$

See [1]. The *Szegö sequences* and *Nehari sequences* defined in the previous sequences are admissible. More precisely, a Nehari sequence is Δ-admissible for the sequence (Δ_n) defined by

$$\Delta_n = \begin{pmatrix} \langle (I-\Gamma_n\Gamma_n^*)^{-1}e, e\rangle & 0 \\ 0 & \langle (I-\Gamma_n^*\Gamma_n)^{-1}e, e\rangle \end{pmatrix}^{-1}, \quad n = 0, 1, \ldots. \tag{2.3}$$

See [1, Theorem 1.10]. We note that a Nehari sequence is not normalized, but that

$$\lim_{n\to\infty} \Delta_n = I_{2p}.$$

A Szegö sequence is admissible, with associated sequence of diagonal matrices given by

$$\Delta_n = \begin{pmatrix} \gamma_{nn}^{(n)} & 0 \\ 0 & \gamma_{00}^{(n)} \end{pmatrix}, \quad n = 0, 1, 2, \ldots. \tag{2.4}$$

See [7, equation (13.19) p. 127]. In particular $\Delta_0 = I_{2p}$ when one fixes $r_0 = I_p$, and then the Szegö sequence is normalized.

Given a Δ-admissible sequence, the system of equations

$$X_n(z) = \begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p \end{pmatrix}^* \begin{pmatrix} zI_p & 0 \\ 0 & I_p \end{pmatrix} X_{n-1}(z), \quad n = 1, 2, \ldots \tag{2.5}$$

is called the *first-order discrete system* defined by the admissible sequence. Associated to (2.5) are a number of functions of the variable z. Direct problems consist in computing these functions from the sequence while inverse problems go the other way around. Inverse problems were studied in [6] in the scalar case and in [2] in the matrix-valued case. Some of the results in [1] are used in the proof of Theorem 1.2, and we review them in the present section. We make the following assumptions: the admissible sequence is normalized (that is, $\Delta_0 = I_{2p}$) and it holds moreover that

$$\sum_{n=0}^{\infty} (\|\alpha_n\| + \|\beta_n\|) < \infty. \tag{2.6}$$

We also assume that the limit

$$\lim_{n\to\infty} \Delta_n = \Delta_\infty \tag{2.7}$$

exists and is strictly positive. We will write

$$\Delta_\infty = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}.$$

These conditions are in particular in force in the rational case.

Assuming (2.6) and (2.7) we proved in [1] that the first-order discrete system (2.5) has a unique solution $X_n(z)$ such that

$$\lim_{n\to\infty} \begin{pmatrix} z^{-n}I_p & 0 \\ o & I_p \end{pmatrix} X_n(z) = \begin{pmatrix} I_p & 0 \\ 0 & I_p \end{pmatrix}, \quad |z| = 1,$$

holds. The *asymptotic equivalence matrix function* is the function $Y(z)$ such that

$$X_n(z) = M_n(z)Y(z)^{-1}$$

where $M_n(z)$ is the solution to (2.5) subject to the initial condition $M_0(z) = I_{2p}$.

The system (2.5) has a unique $\mathbb{C}^{2p\times p}$-valued solution $Y_n(z)$ with the following properties:

(a) $\begin{pmatrix} I_p & -I_p \end{pmatrix} Y_0(z) = 0$, and

(b) $\begin{pmatrix} 0 & I_p \end{pmatrix} Y_n(z) = I_p + o(n), \quad |z| = 1.$

It then holds that

$$\begin{pmatrix} I_p & 0 \end{pmatrix} Y_n(z) = z^n s(z) + o(n)$$

where

$$s(z) = (Y_{11}(z) + Y_{12}(z))(Y_{21}(z) + Y_{22}(z))^{-1}.$$

By analogy with the continuous case, the function $s(z)$ is called the scattering function. It satisfies

$$s(z)^*\delta_1 s(z) = \delta_2, \quad |z| = 1.$$

The function $s_+(z) = (Y_{21}(z)+Y_{22}(z))^{-1}$ belongs to $\mathcal{W}_+^{p\times p}$ and is invertible there. The function $s_-(z) = (Y_{11}(z) + Y_{12}(z))$ belongs to $\mathcal{W}_-^{p\times p}$ and is invertible there. Moreover, $s_+(0) = s_-(\infty) = I_p$. We refer to [1, §2.5] for these facts.

The weight function (which we called in [1] and [2] the spectral function) is defined by

$$w(1/z) = s_-(z)^{-1}\delta_1^{-1}s_-(z)^{-*} = s_+(z)\delta_2^{-1}s_+(z)^*, \quad |z| = 1. \tag{2.8}$$

The weight function is thus strictly positive on the unit circle. Conversely we have the following result, proved in [2].

Theorem 2.1. *Let $w(z) \in \mathcal{W}^{p\times p}$ taking strictly positive values on the unit circle and satisfying the normalization $\frac{1}{2\pi}\int_0^{2\pi} w(e^{it})dt = I_p$. Then, $w(z)$ is the weight function of a uniquely defined first-order discrete system with normalized admissible sequence.*

In the case of the system associated to a Nehari sequence, and when we are in the rational case, that is, when the matrices γ_j admit the representation

$$\gamma_{-j} = ca^j b, \quad j = 0, 1, \ldots$$

for matrices a, b and c of appropriate sizes, explicit formulas are available for the solutions $X_n(z)$ and $M_n(z)$, see [1]. Furthermore, let

$$\alpha_n(z) = a_{n0} + a_{n1}z^{-1} + \cdots \tag{2.9}$$

$$\beta_n(z) = b_{n0} + b_{n1}z^{-1} + \cdots, \tag{2.10}$$

$$\gamma_n(z) = c_{n0} + c_{n1}z + \cdots, \tag{2.11}$$

$$\delta_n(z) = d_{n0} + d_{n1}z + \cdots, \tag{2.12}$$

where the coefficients $a_{nj}, \ldots$ have been defined in (1.6)–(1.7).

The functions (2.9)–(2.12) are defined in [9, p. 964] in the setting of the Nehari interpolation problem. Recall that, with

$$H_n(z) = \begin{pmatrix} \alpha_n(z) & \beta_n(z) \\ \gamma_n(z) & \delta_n(z) \end{pmatrix},$$

the matrix functions $H_n(z)$ satisfy the recursion

$$\begin{pmatrix} I_p & 0 \\ 0 & zI_p \end{pmatrix} H_{n+1}(z) = H_n(z) \begin{pmatrix} I_p & \rho_n \\ \nu_n & I_p \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & zI_p \end{pmatrix}. \tag{2.13}$$

The matrix functions

$$X_n(z) = \begin{pmatrix} I_p & 0 \\ 0 & zI_p \end{pmatrix} H_n(1/\overline{z})^* \begin{pmatrix} z^n I_p & 0 \\ 0 & I_p \end{pmatrix}$$

satisfy the recursion

$$X_{n+1}(z) = \begin{pmatrix} I_p & \rho_n \\ \nu_n & I_p \end{pmatrix}^* \begin{pmatrix} zI_p & 0 \\ 0 & I_p \end{pmatrix} X_n(z), \quad n = 0, 1, \ldots. \tag{2.14}$$

See [2, Theorem 1.10].

3. Proof of Theorem 1.2

The starting point in the proof of Theorem 1.2 is a $\mathbb{C}^{p\times p}$-valued rational function $w(z)$ strictly positive on the unit circle such that

$$\frac{1}{2\pi}\int_0^{2\pi} w(e^{it})dt = I_p.$$

We build the Szegö sequence associated to $w(z)$ and the corresponding first-order discrete system. Both the existence of the limits

$$\lim_{n\to\infty} \gamma_{nn}^{(n)} \quad \text{and} \quad \lim_{n\to\infty} \gamma_{00}^{(n)},$$

and the fact that they are strictly positive were proved in [2]. By hypothesis, these limits are equal:

$$\delta_1 = \lim_{n\to\infty} \gamma_{nn}^{(n)} = \lim_{n\to\infty} \gamma_{00}^{(n)} = \delta_2.$$

Furthermore $w(z)$ can be written as (2.8), where now $\delta_1 = \delta_2$, that is:

$$\begin{aligned} w(z) &= \left((Y_{21} + Y_{22})(\overline{z})\right)^{-1} \delta_1^{-1} \left((Y_{21} + Y_{22})(\overline{z})\right)^{-*} \\ &= \left((Y_{11} + Y_{12})(1/z)\right)^{-1} \delta_1^{-1} \left((Y_{11} + Y_{12})(1/z)\right)^{-*}, \end{aligned} \tag{3.1}$$

where $Y(z) = (Y_{\ell j}(z))_{\ell,j=1,2}$ is the asymptotic equivalence matrix function associated to the first-order system associated to the Szegö sequence; this follows from [1, §2.5] and from the uniqueness of the weight function associated to a given normalized admissible sequence; see [2] for the latter. We then proceed as follows. First we build a function ($z\sigma(z)$, defined by (3.2) below) which takes unitary values on the unit circle, and consider it as a unitary solution of an underlying Nehari extension problem. The spectral factors of $w(z)$ and $w(z)$ itself are then expressed in terms of the functions $\alpha_0(z), \beta_0(z), \gamma_0(z)$ and $\delta_0(z)$ associated to the Nehari problem; see formula (3.8). We consider the normalized Nehari sequence associated to $-z\sigma(z)$ and compute the associated asymptotic equivalence matrix function $Y_N(z)$ (formula (3.11)). This allows us to compute the related weight function $w_N(z)$; see formula (3.9). We find that $w_N(z) = w(z)$, and we conclude by applying the uniqueness result we proved in [2].

We now explicit the strategy outlined above in a number of steps.

STEP 1. *Let* $s(z)$ *be the scattering function associated to* $w(z)$ *and let*

$$\sigma(z) = \sqrt{\delta_1} s(z) \sqrt{\delta_1}^{-1}.$$

Set

$$z\sigma(z) = \sum_{\mathbb{Z}} \gamma_\ell z^\ell. \tag{3.2}$$

Then

$$z\sigma(z) = (\alpha_0(z)z + \beta_0(z))(z\gamma_0(z) + \delta_0(z))^{-1} \tag{3.3}$$

where the functions $\alpha_0(z), \beta_0(z), \gamma_0(z)$ *and* $\delta_0(z)$ *are built from* Γ *as in* (2.9)–(2.12) *with* $n = 0$.

Indeed, consider the Nehari extension problem associated to the sequence

$$\gamma_0, \gamma_{-1}, \gamma_{-2}, \ldots.$$

The function $z\sigma(z)$ is a solution of this problem. It is unitary on the unit circle. It admits a Wiener–Hopf factorization (with indices) since $s(z)$ admits a Wiener–Hopf factorization. To obtain (3.3) we use an approximation argument. We consider the function $\epsilon z\sigma(z)$ where $0 < \epsilon < 1$, and consider the Nehari extension problem associated to the sequence

$$\epsilon\gamma_0, \epsilon\gamma_{-1}, \epsilon\gamma_{-2}, \ldots.$$

By [9, XXXV.4] there exist functions $z \mapsto R(z, \epsilon) \in \mathcal{W}_+^{p\times p}$, and taking strictly contractive values on the unit circle such that

$$\begin{aligned} \epsilon z\sigma(z) = (&\alpha_0(z,\epsilon)a_{00}^{-1/2}(\epsilon)zR(z,\epsilon) + \beta_0(z,\epsilon)d_{00}(\epsilon)^{-1/2})\times \\ &\times (\gamma_0(z,\epsilon)a_{00}(\epsilon)^{-1/2}zR(z,\epsilon) + \delta_0(z,\epsilon)d_{00}(\epsilon)^{-1/2})^{-1}, \end{aligned} \tag{3.4}$$

where the functions $\alpha_0(z, \epsilon), \ldots$ are built from (2.9)–(2.12) with $\epsilon\Gamma$ in place of Γ. Let $(a_{0j}(\epsilon))_{j=0,1,\ldots} = (I - \epsilon^2\Gamma\Gamma^*)^{-1}e$. We have:

$$\begin{aligned} \|(a_{0j}) - (a_{0j}(\epsilon))\|_{\ell_2^{p\times p}} &= \|(I - \Gamma\Gamma^*)^{-1}(I - \epsilon^2\Gamma\Gamma^*)^{-1}(1 - \epsilon^2)\Gamma\Gamma^* e\|_{\ell_2^{p\times p}} \\ &\le (1 - \epsilon^2)\frac{1}{(1 - \|\Gamma\|^2)^2}\|\Gamma\|^2. \end{aligned}$$

We conclude that

$$\lim_{\epsilon \to 0} a_{00}(\epsilon) = a_{00},$$

and that $\alpha_0(z,\epsilon) \to \alpha_0(z)$ point-wise for every z in the open unit disk $\mathbb{D}$ as $\epsilon \to 0$. Similarly,

$$\lim_{\epsilon \to 0} d_{00}(\epsilon) = d_{00},$$

and $\beta_0(z,\epsilon), \gamma_0(z,\epsilon)$ and $\delta_0(z,\epsilon)$ tend point-wise to $\beta_0(z), \gamma_0(z)$ and $\delta_0(z)$ respectively as $\epsilon \to 0$ for $z \in \mathbb{D}$. It follows from (3.4) that $R(z,\epsilon)$ tends point-wise to a function $R(z)$. This function is rational since the functions $\alpha_0(z), \ldots$ are rational. Its takes unitary values on $\mathbb{T}$ since $\sigma(z)$ does, and since $H_0(z)U_0^{-1}$ takes J-unitary values on $\mathbb{T}$, where we have set

$$U_0 = \begin{pmatrix} a_{00}^{1/2} & 0 \\ 0 & d_{00}^{1/2} \end{pmatrix}. \tag{3.5}$$

It is therefore a finite matrix-valued Blaschke product, and in particular belongs to $\mathcal{W}_+^{p\times p}$, and its inverse belongs to $\mathcal{W}_-^{p\times p}$. We recall that

$$d_{00}^{-1/2}\delta_0^{-1}(z) \quad \text{and} \quad d_{00}^{-1/2}\delta_0^{-1}(z)\gamma_0(z)a_{00}^{1/2}$$

belong to $\mathcal{W}_+^{p\times p}$ and that

$$\|d_{00}^{-1/2}\delta_0^{-1}(z)\gamma_0(z)a_{00}^{1/2}\| < 1 \quad \text{for} \quad |z| = 1.$$

Similarly

$$a_{00}^{-1/2}\alpha_0^{-1}(z) \quad \text{and} \quad a_{00}^{-1/2}\alpha_0^{-1}(z)\beta_0(z)d_{00}^{1/2}$$

belong to $\mathcal{W}_-^{p\times p}$ and

$$\|a_{00}^{-1/2}\alpha_0^{-1}(z)\beta_0(z)d_{00}^{1/2}\| < 1 \quad \text{for} \quad |z| = 1.$$

See [9, p. 959] for the claims on the invertibility of $\alpha_0(z)$ and $\delta_0(z)$. The other claims follow from the fact that the function $H_0(z)U_0^{-1}$ is J-unitary on the unit circle; see [1, Proof of Theorem 1.10, STEP 2].

Using these facts and the equality

$$\sigma(z) = (\alpha_0(z)a_{00}^{-1/2} + \beta_0(z)d_{00}^{-1/2}(zR(z))^{-1})R(z)\times \\ \times (\gamma_0(z)a_{00}^{-1/2}zR(z) + \delta_0(z)d_{00}^{-1/2})^{-1},$$

we see that $\sigma(z)$ has a Wiener–Hopf factorization if and only if $R(z)$ is a unitary constant, say R. We now show that $R = I_p$. Indeed, by uniqueness of the spectral factorization there exists an invertible matrix X such that

$$\sqrt{\delta_1}s_-(z) = (\alpha_0(z)a_{00}^{-1/2}R + \frac{\beta_0(z)}{z}d_{00}^{-1/2})X$$
$$s_+(z)\sqrt{\delta_1}^{-1} = X^{-1}(\gamma_0(z)a_{00}^{-1/2}Rz + \delta_0(z)d_{00}^{-1/2})^{-1}.$$

In particular

$$\begin{aligned} \sqrt{\delta_1} &= \alpha_0(\infty)a_{00}^{-1/2}RX \\ \sqrt{\delta_1}^{-1} &= X^{-1}d_{00}^{1/2}\delta_0(0)^{-1}. \end{aligned} \tag{3.6}$$

But $\alpha_0(\infty) = a_{00}$ and $\delta_0(0) = d_{00}$ (see (2.9) and (2.12)). By hypothesis $a_{00} = d_{00}$, and so multiplying the above equalities side by side we obtain

$$I_p = X^{-1}RX,$$

and so $R = I_p$. This ends the proof of Step 1. We note that the hypothesis $\delta_1 = \delta_2$ and $a_{00} = d_{00}$ was used to proceed from (3.6) and obtain $R = I_p$.

From (3.3) we see that

$$\begin{aligned} s(z) &= \sqrt{\delta_1}^{-1}\left(\alpha_0(z)a_{00}^{-1/2} + \frac{\beta_0(z)}{z}a_{00}^{-1/2}\right)\left(\gamma_0(z)a_{00}^{-1/2}z + \delta_0(z)a_{00}^{-1/2}\right)^{-1}\sqrt{\delta_1} \\ &= \sqrt{\delta_1}^{-1}\left(\alpha_0(z) + \frac{\beta_0(z)}{z}\right)(\gamma_0(z)z + \delta_0(z))^{-1}\sqrt{\delta_1}. \end{aligned}$$

This last expression can be written in turn as:

$$\left(\sqrt{\delta_1}^{-1}\left(\alpha_0(z)+\frac{\beta_0(z)}{z}\right)\alpha_0(\infty)^{-1}\sqrt{\delta_1}\right)\left(\sqrt{\delta_1}^{-1}(\gamma_0(z)z+\delta_0(z))\delta_0(0)^{-1}\sqrt{\delta_1}\right)^{-1}$$

(recall that $\alpha_0(\infty) = \delta_0(0)$) and so, in view of the normalizing conditions

$$s_-(\infty) = s_+(0) = I_p,$$

we have that:

$$\begin{aligned} s_-(z) &= \sqrt{\delta_1}^{-1}\left(\alpha_0(z) + \frac{\beta_0(z)}{z}\right)(\alpha_0(\infty))^{-1}\sqrt{\delta_1} \\ s_+(z) &= \left(\sqrt{\delta_1}^{-1}(\gamma_0(z)z + \delta_0(z))\,\delta_0(0)^{-1}\sqrt{\delta_1}\right)^{-1}. \end{aligned} \tag{3.7}$$

Step 2. *The weight function associated to the Szegö sequence is given by:*

$$\begin{aligned} w(1/z) &= \sqrt{\delta_1}^{-1}\alpha_0(\infty)\times \\ &\times\left(\left(\alpha_0(z) + \frac{\beta_0(z)}{z}\right)^{-1}\left(\alpha_0(z) + \frac{\beta_0(z)}{z}\right)^{-*}\right)\alpha_0(\infty)\sqrt{\delta_1}^{-1}. \end{aligned} \tag{3.8}$$

Indeed, from formulas (2.8) and (3.7) we have:

$$\begin{aligned} w(1/z) = &\left(\sqrt{\delta_1}^{-1}\left(\alpha_0(z) + \frac{\beta_0(z)}{z}\right)\alpha_0(\infty)^{-1}\sqrt{\delta_1}\right)^{-1}\delta_1^{-1}\times \\ &\times\left(\sqrt{\delta_1}^{-1}\left(\alpha_0(z) + \frac{\beta_0(z)}{z}\right)\alpha_0(\infty)^{-1}\sqrt{\delta_1}\right)^{-*} \end{aligned}$$

and hence the result.

Step 3. *Assume that the pair (α_n, β_n), $n = 1, 2, \ldots$ is Δ-admissible. Then the sequence $(-\alpha_n, -\beta_n)$, $n = 1, 2, \ldots$ is also admissible, with the same sequence Δ. In particular the pair $(-\rho_{n-1}, -\nu_{n-1})$, $n = 1, 2, \ldots$ is admissible.*

Indeed, from (2.1) we have

$$J\begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p \end{pmatrix}JJJ\Delta_nJJ\begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p \end{pmatrix}^* J = JJ\Delta_{n-1}J, \quad n = 1, 2, \ldots$$

and so

$$\begin{pmatrix} I_p & -\alpha_n \\ -\beta_n & I_p \end{pmatrix} J\Delta_n \begin{pmatrix} I_p & -\alpha_n \\ -\beta_n & I_p \end{pmatrix}^* = J\Delta_{n-1}, \quad n = 1, 2, \ldots$$

Step 4. *The weight function $w_N(z)$ associated to the pair*

$$(-a_{00}^{1/2}\rho_{n-1}a_{00}^{-1/2}, -a_{00}^{1/2}\nu_{n-1}a_{00}^{-1/2})$$

is given by the formula

$$w_N(1/z) = a_{00}^{1/2}\left(\alpha_0(z) + \frac{\beta_0(z)}{z}\right)^{-1}\left(\alpha_0(z) + \frac{\beta_0(z)}{z}\right)^{-*} a_{00}^{1/2}. \tag{3.9}$$

Indeed, when one replaces σ by $-\sigma$ the Hankel operator Γ defined by (1.4) with $n = 0$ and built from the Fourier coefficients of σ is replaced by $-\Gamma$. The sequence

$$(-a_{00}^{1/2}\rho_{n-1}a_{00}^{-1/2}, -a_{00}^{1/2}\nu_{n-1}a_{00}^{-1/2}) \tag{3.10}$$

is an admissible normalized pair, with associated sequence of diagonal

$$\Delta_0^{-1/2}\Delta_n\Delta_0^{-1/2},$$

with Δ_n given by (2.3).

We now consider the first-order discrete system associated to (3.10). The recursion (2.14) needs to be changed as follows: the minus sign in Γ implies that the functions $\beta_0(z)$ and $\gamma_0(z)$ are replaced by their opposite, and thus $H_n(z)$ by $JH_n(z)J$. Finally, $X_n(z)$ is replaced by $a_{00}^{1/2}X_n(z)a_{00}^{-1/2}$ since we consider the normalized pair (3.10). Hence the solution to the corresponding first-order discrete system with value I_{2p} at $n = 0$ is equal to

$$M_n(z) = a_{00}^{1/2}\begin{pmatrix} I_p & 0 \\ 0 & zI_p \end{pmatrix} JH_n(1/\overline{z})^*J \begin{pmatrix} z^n & 0 \\ 0 & I_p \end{pmatrix} JH_0(1/\overline{z})^{-*}J \begin{pmatrix} I_p & 0 \\ 0 & z^{-1}I_p \end{pmatrix} a_{00}^{-1/2}$$

and the solution subject to the condition

$$\lim_{n\to\infty}\begin{pmatrix} z^{-n}I_p & 0 \\ 0 & I_p \end{pmatrix} X_n(z) = \begin{pmatrix} I_p & 0 \\ 0 & I_p \end{pmatrix}$$

is equal to

$$X_n(z) = a_{00}^{1/2}\begin{pmatrix} I_p & 0 \\ 0 & zI_p \end{pmatrix} JH_n(1/\overline{z})^*J \begin{pmatrix} z^n & 0 \\ 0 & I_p \end{pmatrix}\begin{pmatrix} I_p & 0 \\ 0 & z^{-1}I_p \end{pmatrix} a_{00}^{-1/2}.$$

Let $Y_N(z)$ be the associated asymptotic equivalence matrix function. Since $X_n(z) = M_n(z)Y_N(z)^{-1}$ we deduce that $Y_N(z)$ is equal to

$$Y_N(z) = a_{00}^{1/2}\begin{pmatrix} I_p & 0 \\ 0 & zI_p \end{pmatrix} JH_0(1/\overline{z})^{-*}J \begin{pmatrix} I & 0 \\ 0 & z^{-1}I_p \end{pmatrix} a_{00}^{-1/2}. \tag{3.11}$$

By [1, (2.4)],

$$Y_N(z)JY_N(z)^* = a_{00}J, \quad |z| = 1.$$

Thus,

$$\begin{aligned}
Y_N(z) &= a_{00} J Y(1/\overline{z})^{-*} J \\
&= a_{00} a_{00}^{-1/2} J J \begin{pmatrix} I_p & 0 \\ 0 & z I_p \end{pmatrix} H_0(z) J \begin{pmatrix} I_p & 0 \\ 0 & z^{-1} I_p \end{pmatrix} a_{00}^{-1/2} J \\
&= a_{00}^{1/2} \begin{pmatrix} I_p & 0 \\ 0 & z I_p \end{pmatrix} H_0(z) \begin{pmatrix} I_p & 0 \\ 0 & z^{-1} I_p \end{pmatrix} a_{00}^{-1/2} \\
&= a_{00}^{1/2} \begin{pmatrix} \alpha_0(z) & \frac{\beta_0(z)}{z} \\ z\gamma_0(z) & \delta_0(z) \end{pmatrix} a_{00}^{-1/2}.
\end{aligned}$$

Using formula (3.1) we obtain that the corresponding weight function is given by (3.9). To that purpose note that the matrices δ_1 and δ_2 in formula (3.1) are here equal to the diagonal block entries of

$$\lim_{n\to\infty} U_0^{-1} U_n^2 U_0^{-1} = \begin{pmatrix} a_{00} & 0 \\ 0 & a_{00} \end{pmatrix},$$

where U_n is defined by

$$U_n = \begin{pmatrix} a_{nn}^{1/2} & 0 \\ 0 & d_{nn}^{1/2} \end{pmatrix}, \quad n = 0, 1, \ldots .$$

Step 5. *It holds that* $w_N(z) = w_{Sz}(z)$.

Indeed, it follows from formulas (3.8) and (3.9) that

$$w_N(z) = Q w_{Sz}(z) Q^*$$

with $Q = a_{00}^{1/2} \alpha_0(\infty)^{-1} \sqrt{\delta_1}$. The normalization

$$\frac{1}{2\pi} \int_0^{2\pi} w_N(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} w_{Sz}(e^{it}) dt = I_p$$

forces Q to be unitary. Since $\alpha_0(\infty) = a_{00}$ we have that

$$\sqrt{a_{00}}^{-1} \sqrt{\delta_1} = Q,$$

so that $\sqrt{\delta_1} = \sqrt{a_{00}} Q$. By uniqueness of the polar decomposition we obtain $Q = I_p$.

Step 6. *The Szegö sequence and the Nehari normalized sequence are related by*

$$\begin{aligned}
\alpha_n &= -a_{00}^{1/2} \rho_{n-1} a_{00}^{-1/2} \\
\beta_n &= -a_{00}^{1/2} \nu_{n-1} a_{00}^{-1/2}, \quad n = 1, 2, \ldots .
\end{aligned}$$

This follows from the uniqueness of the normalized admissible sequence associated to a normalized weight function.

4. An example

We now present an example where (1.8) is in force. As already mentioned, we showed in the scalar case (that is, in the case $p=1$) that $\alpha_n=\overline{\beta_n}$ and that the diagonals Δ_n are scalar matrices; see (2.2) and [1]. Another example is when $w(z)$ satisfies the symmetries

$$w(1/z)=w(z) \quad \text{and} \quad \overline{w(\overline{z})}=w(z). \tag{4.1}$$

Thus, the Fourier coefficients of $w(z)$ satisfy

$$r_n=r_{-n}=\overline{r_n}=r_n^t$$

where the superscript t denotes transpose, and we can write:

$$w(e^{it})=I_p+2\sum_1^\infty r_n\cos(nt).$$

We prove that the Szegö sequence and the Nehari sequence satisfy (1.8) when the conditions in (4.1) are in force. We denote by ι the $\mathbb{C}^{(n+1)p\times(n+1)p}$ block matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & I_p \\ 0 & 0 & \cdots & I_p & 0 \\ & & & & \\ 0 & I_p & \cdots & & 0 \\ I_p & 0 & \cdots & & 0 \end{pmatrix}.$$

Since $r_n=r_n^t$ we have that $T_n=\iota T_n\iota$. It follows (see also [1, §1.2]) that

$$\gamma_{n-i,n-j}^{(n)}=\gamma_{ij}^{(n)}.$$

In particular we have

$$\gamma_{00}^{(n)}=\gamma_{nn}^{(n)} \quad \text{and} \quad \gamma_{0n}^{(n)}=\gamma_{n0}^{(n)}.$$

Equation (2.4) implies that the Szegö sequence satisfies (1.8). Equation (1.5) shows moreover $\alpha_n=\beta_n$.

We now turn to the Nehari sequence. Condition (4.1) implies that $w(z)$ can be factorized as

$$w(z)=\sigma_+(z)\sigma_+(1/z)^t$$

where $\sigma_+(z)$ and its inverse are in $\mathcal{W}_+^{p\times p}$ and moreover $\sigma_+(z)$ is real:

$$\sigma_+(\overline{z})=\overline{\sigma_+(z)}.$$

Indeed, we have the factorizations

$$w(z)=g_+(z)g_+(1/\overline{z})^*=\overline{g_+(\overline{z})}g_+(1/z)^t$$

where $g_+(z)$ and its inverse belong to $\mathcal{W}_+^{p\times p}$. Thus there exists a unitary matrix U such that

$$g_+(z)=\overline{g_+(\overline{z})}U.$$

Setting $z = 1$ in this equation we obtain

$$U = \overline{g_+(1)^{-1}} g_+(1)$$

and so

$$g_+(z)g_+(1)^{-1} = \overline{g_+(\overline{z})g_+(1)^{-1}}.$$

Thus the function $g_+(z)g_+(1)^{-1}$ is real. We have

$$w(z) = g_+(z)g_+(1)^{-1}w(1)(g_+(1/z)g_+(1)^{-1})^t.$$

The matrix $w(1)$ is real and positive; it is thus symmetric positive and as such has a real square root (write $w(1) = QDQ^t$ where Q is orthogonal and D is a diagonal with positive entries and take as square root $QD^{1/2}Q^t$). The function

$$\sigma_+(z) = g_+(z)g_+(1)^{-1}w(1)^{1/2}$$

is then real and is a spectral factor of $w(z)$. Since $w(z) = w(1/z)$ we can take $\sigma_-(z) = (\sigma_+(1/z))^{-1}$ as spectral factor in $\mathcal{W}_-^{p\times p}$. Let

$$s(z) = (\sigma_+(1/z))^{-1}\sigma_+(z). \tag{4.2}$$

We claim that

$$s(z) = s(z)^t = \overline{s(\overline{z})}.$$

Indeed, $s(z)$ is real since $\sigma_+(z)$ is real. Moreover we have

$$\sigma_+(z)(\sigma_+(1/z))^t = \sigma_+(1/z)(\sigma_+(z))^t,$$

and hence

$$s(z) = (\sigma_+(z))^t(\sigma_+(1/z))^{-t}, \quad \text{that is} \quad s(z) = s(z)^t.$$

It follows that in the representation

$$zs(z) = \sum_{\mathbb{Z}} \gamma_k z^k$$

the matrices γ_k are real and symmetric, and thus the associated Hankel operators Γ_n are self-adjoint. It follows from formula (2.3) that the Nehari sequence satisfies (1.8).

We note that the requirements in (4.1) appear in Krein's work in [10] in the continuous case. We also note that functions of the form (4.2) were considered in [3, p. 141].

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Multivariable Generalizations of the Schur Class: Positive Kernel Characterization and Transfer Function Realization

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Abstract. The operator-valued Schur class is defined to be the set of holomorphic functions S mapping the unit disk into the space of contraction operators between two Hilbert spaces. There are a number of alternate characterizations: the operator of multiplication by S defines a contraction operator between two Hardy Hilbert spaces, S satisfies a von Neumann inequality, a certain operator-valued kernel associated with S is positive-definite, and S can be realized as the transfer function of a dissipative (or even conservative) discrete-time linear input/state/output linear system. Various multivariable generalizations of this class have appeared recently, one of the most encompassing being that of Muhly and Solel where the unit disk is replaced by the strict unit ball of the elements of a dual correspondence E^σ associated with a W^*-correspondence E over a W^*-algebra $\mathcal{A}$ together with a $*$-representation σ of $\mathcal{A}$. The main new point which we add here is the introduction of the notion of reproducing kernel Hilbert correspondence and identification of the Muhly-Solel Hardy spaces as reproducing kernel Hilbert correspondences associated with a completely positive analogue of the classical Szegö kernel. In this way we are able to make the analogy between the Muhly-Solel Schur class and the classical Schur class more complete. We also illustrate the theory by specializing it to some well-studied special cases; in some instances there result new kinds of realization theorems.

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1. Introduction

The classical Schur class $\mathcal{S}$ (consisting of holomorphic functions mapping the unit disk $\mathbb{D}$ into the closed unit disk $\overline{\mathbb{D}}$) along with its operator-valued generalization has been an object of intensive study over the past century (see [45] for the original paper of Schur and [26] for a survey of some of the impact and applications in signal processing). To formulate the definition of the operator-valued version, we let $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ denote the space of bounded linear operators acting between Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. We also let $H^2_{\mathcal{U}}(\mathbb{D})$ and $H^2_{\mathcal{Y}}(\mathbb{D})$ be the standard Hardy spaces of $\mathcal{U}$-valued (respectively $\mathcal{Y}$-valued) holomorphic functions on the unit disk $\mathbb{D}$. By the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ we mean the set of $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions holomorphic on the unit disk $\mathbb{D}$ with values $S(z)$ having norm at most 1 for each $z \in \mathbb{D}$. The class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ admits several remarkable characterizations. The following result is well known and is formulated as the prototype for the multivariable generalizations to follow.

Theorem 1.1. *Let S be an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function defined on $\mathbb{D}$. The following are equivalent:*

1. *$S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, i.e., S is analytic on $\mathbb{D}$ with contractive values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$.*

(1′) *The multiplication operator $M_S \colon f(z) \mapsto S(z) \cdot f(z)$ defines a contraction from $H^2_{\mathcal{U}}(\mathbb{D})$ into $H^2_{\mathcal{Y}}(\mathbb{D})$.*

(1″) *S is analytic and satisfies the von Neumann's inequality:* if T is any strictly contractive operator on a Hilbert space $\mathcal{K}$, i.e., $\|T\| < 1$, then $S(T)$ is a contraction operator ($\|S(T)\| \leq 1$), where $S(T)$ is the operator defined by

$$S(T) = \sum_{n=0}^{\infty} S_n \otimes T^n \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K}) \quad \text{if} \quad S(z) = \sum_{n=0}^{\infty} S_n z^n.$$

2. *The function $K_S \colon \mathbb{D} \times \mathbb{D} \to \mathcal{L}(\mathcal{Y})$ given by*
$$K_S(z, w) = \frac{I_{\mathcal{Y}} - S(z)S(w)^*}{1 - z\overline{w}}$$
is a positive kernel on $\mathbb{D} \times \mathbb{D}$.

3. *There exists a Hilbert space $\mathcal{H}$ and a coisometric (or even unitary or contractive) connecting operator (or colligation) $\mathbf{U}$ of the form*
$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix}$$
so that $S(z)$ can be realized in the form
$$S(z) = D + zC(I_{\mathcal{H}} - zA)^{-1}B. \tag{1.1}$$

From the point of view of systems theory, the function (1.1) is the *transfer function* of the linear system

$$\mathbf{\Sigma} = \mathbf{\Sigma}(\mathbf{U}) \colon \left\{ \begin{array}{rcl} x(n+1) & = & Ax(n) + Bu(n) \\ y(n) & = & Cx(n) + Du(n) \end{array} \right. .$$

The following well-known proposition gives several equivalent definitions for the term "positive kernel" used in condition (2) in Theorem 1.1. The scalar case ($\mathcal{Y} = \mathbb{C}$) of this result goes back to the paper of Aronszajn [7], but is also often attributed to E.H. Moore and Kolmogorov, while the vector-valued case has been well exploited in the function-theoretic operator theory literature over the years (see [47, 19]).

Proposition 1.2. *Let $K\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{Y})$ be a given function. Then the following conditions are equivalent:*

1. *For any finite collection of points $\omega_1, \dots, \omega_N \in \Omega$ and of vectors $y_1, \dots, y_N \in \mathcal{Y}$ $(N = 1, 2, \dots)$ it holds that*
$$\sum_{i,j=1,\dots,N} \langle K(\omega_i, \omega_j) y_j, y_i \rangle_{\mathcal{Y}} \geq 0. \tag{1.2}$$
2. *There exists an operator-valued function $H\colon \Omega \to \mathcal{L}(\mathcal{H}, \mathcal{Y})$ for some auxiliary Hilbert space $\mathcal{H}$ so that*
$$K(\omega', \omega) = H(\omega') H(\omega)^*. \tag{1.3}$$
3. *There exists a Hilbert space $\mathcal{H}(K)$ of $\mathcal{Y}$-valued functions f on Ω so that the function $K(\cdot, \omega)y$ is in $\mathcal{H}(K)$ for each $\omega \in \Omega$ and $y \in \mathcal{Y}$ and has the reproducing property*
$$\langle f, K(\cdot, \omega) y \rangle_{\mathcal{H}(K)} = \langle f(\omega), y \rangle_{\mathcal{Y}}.$$

When any (and hence all) of these equivalent conditions hold, we say that K is a positive kernel on $\Omega \times \Omega$.

We provide a sketch of the proof of Theorem 1.1 as a model for how extensions to more general settings may proceed.

Sketch of the proof of Theorem 1.1. The easy part is (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1″) $\Longrightarrow$ (1′) $\Longrightarrow$ (1):

(3) $\Longrightarrow$ (2): Assume that $S(z)$ is as in (1.1) with $\mathbf{U}$ unitary, and hence, in particular, coisometric. From the relations arising from the coisometric property of $\mathbf{U}$:
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$
one can verify that
$$\begin{aligned} I - S(z)S(w)^* &= I - [D + zC(I - zA)^{-1}B][D + wC(I - wA)^{-1}B]^* \\ &= C(I - zA)^{-1}[(1 - z\overline{w})I_{\mathcal{H}}](I - \overline{w}A^*)^{-1}C^*. \end{aligned}$$
This implies that $H(z) = C(I - zA)^{-1}$ satisfies (1.3).

(2) $\Longrightarrow$ (1″): Due to $I - S(z)S(w)^* = H(z)[(1 - z\overline{w})I_{\mathcal{H}}]H(w)^*$, we can see that for any $\|T\| < 1$
$$I - S(T)S(T)^* = H(T)[(1 - TT^*) \otimes I_{\mathcal{H}}]H(T)^* \geq 0.$$

(1″) ⟹ (1′): Observe that $M_S = S(\mathbf{S}) = s - \lim_{r \uparrow 1} S(r\mathbf{S})$ where $\mathbf{S}$ is the shift operator M_z on $H^2(\mathbb{D})$. Thus the fact that $\|S(r\mathbf{S})\| \le 1$ for any $r < 1$ implies $\|M_S\| \le 1$.

(1′) ⟹ (1): Note that since $S(z)u = M_S \cdot u$ for any $u \in \mathcal{U}$, we have $\|M_S\|_{op} = \|S\|_\infty$. So $\|M_S\| \le 1$ implies that $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$.

The harder part is (1) ⟹ (1′) ⟹ (1″) ⟹ (2) ⟹ (3):

(1) ⟹ (1′): We can view $H^2(\mathbb{D}) \subset L^2(\mathbf{T})$. Thus $\|M_S u\|_{L^2(\mathbf{T})} \le \|S\|_\infty \cdot \|u\|_{L^2(\mathbf{T})}$.

(1′) ⟹ (1″): According to the Sz.-Nagy dilation theorem, any contraction operator T has a unitary dilation U. In the strictly contractive case $\|T\| < 1$, one can show that in fact the unitary dilation is the bilateral shift with some multiplicity N: $U = \mathbf{S} \otimes I_N$ (if $N = \infty$, we interpret I_N as the identity operator on ℓ^2). We then have $T^n = P_K(\mathbf{S} \otimes I_N)^n|_{\mathcal{K}}$. Therefore $\|S(T)\| = \|P_{\mathcal{Y}\otimes\mathcal{K}} S(\mathbf{S} \otimes I_N)|_{\mathcal{U}\otimes\mathcal{K}}\| \le \|M_S\| \le 1$.

(1″) ⟹ (2): A direct proof of this implication can be done via a rather long, intricate argument using a Gelfand-Naimark-Segal construction in conjunction with a Hahn-Banach separation argument—we refer to this as a GNS/HB argument. For the polydisk setting, the argument originates in [1]; the version for a general semigroupoid setting in [23] covers in particular the classical setting here.

Alternatively, one can avoid the GNS/HB argument via the following shortcut:

(1″) ⟹ (1′) ⟹ (2): We have seen that (1″) ⟹ (1′) is easy. For (1′) ⟹ (2), we assume $\|M_S\| \le 1$. View $H^2(\mathbb{D})$ as the reproducing kernel Hilbert space $\mathcal{H}(k_{\mathrm{Sz}})$, where $k_{\mathrm{Sz}}(z,w) = \frac{1}{1-z\overline{w}}$ is the Szegö kernel. Since $M_S^* k_{S_z}(\cdot, w)y = k_{S_z}(\cdot, w)S(w)^*y$, we see that

$$\sum_{i,j=1,\dots,N} \langle K_S(z_i, z_j)y_j, y_i\rangle_{\mathcal{Y}} = \|\sum_j k_{\mathrm{Sz}}(\cdot, z_j)y_j\|^2 - \|(M_S)^* \sum_j k_{\mathrm{Sz}}(\cdot, z_j)y_j\|^2 \ge 0$$

and it follows (via criterion (1.2)) that K_S is a positive kernel on $\mathbb{D} \times \mathbb{D}$.

(2) ⟹ (3): This implication can be done by the now standard lurking isometry argument – see [8] where this coinage was introduced. □

The purpose of this paper is to study recent extensions of the Schur class and the associated analogues of Theorem 1.1 to more general multivariable settings. In Section 2 we describe two such extensions: the Drury-Arveson space setting and the free-semigroup setting. We emphasize how all the ingredients of the proof of Theorem 1.1 sketched above have direct analogues in these two settings; hence the proof of the analogues of Theorem 1.1 for these two settings (see Theorem 2.1 and Theorem 2.3 below) directly parallel the proof of Theorem 1.1 as sketched above. A far more sophisticated generalized Schur class has been introduced by Muhly and Solel (see [33, 36]). The main contribution of the present paper is to introduce the notion of reproducing kernel Hilbert correspondence and an analogue of the Fourier (or Z-) transform for the Muhly-Solel setting. The starting point for most of the constructions is a W^*-correspondence E over a W^*-algebra $\mathcal{A}$ together with a $*$-representation σ of $\mathcal{A}$. We show that the image, denoted in our notation as $H^2(E,\sigma)$ which is an analogue of H^2, of a Muhly-Solel Fock space, denoted as $\mathcal{F}^2(E,\sigma)$ in our notation which is an analogue of $\ell^2(\mathbb{Z}_+)$, under this Z-transform is a space of $\mathcal{E}$-valued functions ($\mathcal{E}$ equal to a coefficient Hilbert space) on the Muhly-Solel generalized unit disk $\mathbb{D}((E^\sigma)^*)$[1] and that an element S of the Muhly-Solel Schur class as introduced in [36] induces a bounded multiplication operator on $H^2(E,\sigma)$. We also obtain analogues of the other parts of Theorem 1.1 for this setting (see Theorem 5.1 in Section 5 below) and thus obtain a more complete analogy between the Muhly-Solel Schur class and the classical Schur class than that presented in [36]. Section 3 develops required preliminaries concerning general correspondences, including the notions of *reproducing kernel correspondence* and of *reproducing kernel Hilbert correspondence*; these are natural elaborations of the Kolmogorov decomposition for a completely positive kernel found in [18]. Section 4 introduces the spaces $H^2(E,\sigma)$ and $H^\infty(E,\sigma)$ which are the analogues of the Hardy spaces H^2 and H^∞ for this setting. The final section 6 applies the general theory to some familiar more concrete special cases. Specifically we make explicit how the classical case discussed above as well as the Drury-Arveson setting and the free-semigroup algebra setting discussed in Section 2 are particular cases of the Muhly-Solel setting. The general theory here leads to more structured versions of these well-studied settings and corresponding new types of realization theorems. We also discuss one of the main examples motivating the work in [31, 33, 36], namely the setting of analytic crossed-product algebras. It is interesting to note that the realization theorem for a particular instance of this example amounts to the realization theorem for input-output maps of conservative time-varying linear systems obtained in [4].

Another class of examples covered by the Muhly-Solel setting are graph algebras (also known as semigroupoid algebras) [30, 32, 27]; we do not discuss these here. There are still other types of generalized Schur classes which are not subsumed under the Muhly-Solel Fock space/correspondence setup. We mention the Schur-Agler class for the polydisk (see [1, 2, 14] and for more general domains [5, 9]), the noncommutative Schur-Agler class (see [12, 13]), and higher-rank graph algebras (see [28]). A differentiating feature of these variants of the Schur class is a more implicit version of condition (2) in Theorem 1.1 where the single positive kernel (the Szegö kernel $\frac{1}{1-z\overline{w}}$) is replaced by a whole family of positive kernels. An abstract framework using this feature as the point of departure is the semigroupoid approach of Dritschel-Marcantognini-McCullough [23] which incorporates all the aforementioned settings in [1, 2, 14, 12, 28]. However the theory in [23] does not appear to include the analytic crossed-product algebras included in the Muhly-Solel scheme since it does not allow for the action of a W^*-algebra $\mathcal{A}$ acting on

[1] In nice cases, the general situation collapses to this statement; more correctly, the vector-valued functions are defined on $\mathbb{D}((E^\sigma)^*) \times \sigma(\mathcal{A})'$ where $\sigma(\mathcal{A})'$ is the commutant of the image $\sigma(\mathcal{A})$ of σ in $\mathcal{L}(\mathcal{E})$.

the ambient Hilbert space. It is conceivable that some sort of synthesis of these two disparate approaches is possible; the recent work on product decompositions over general semigroups (see [46]) appears to be a start in this direction.

The notation is mostly standard but we mention here a few conventions for reference. For Ω any index set, $\ell^2(\Omega)$ denotes the space of complex-valued functions on Ω which are absolutely square summable:

$$\ell^2(\Omega) = \{\xi\colon \Omega \to \mathbb{C}\colon \sum_{\omega\in\Omega} |\xi(\omega)|^2 < \infty\}.$$

Most often the choice $\Omega = \mathbb{Z}$ (the integers) or $\Omega = \mathbb{Z}_+$ (the nonnegative integers) appears. For $\mathcal{H}$ a Hilbert space, we use $\ell^2_{\mathcal{H}}(\Omega)$ as shorthand for $\ell^2(\Omega) \otimes \mathcal{H}$ (the space of $\mathcal{H}$-valued function on Ω square-summable in norm). More general versions where $\mathcal{H}$ may be a correspondence also come up from time to time.

2. Some multivariable Schur classes

In this section we introduce two multivariable settings (the Drury-Arveson space setting and the free semigroup algebra setting) for the Schur class and formulate the analogue of Theorem 1.1 for these two settings.

2.1. Drury-Arveson space

A multivariable generalization of the Szegö kernel $k(z,w) = (1 - z\bar{w})^{-1}$ much studied of late is the positive kernel

$$k_d(z,w) = \frac{1}{1 - \langle z, w\rangle} \text{ on } \mathbb{B}^d \times \mathbb{B}^d,$$

where $\mathbb{B}^d = \left\{ z = (z_1, \dots, z_d) \in \mathbb{C}^d \colon \langle z, z\rangle < 1 \right\}$ is the unit ball of the d-dimensional Euclidean space $\mathbb{C}^d$. By $\langle z, w\rangle = \sum_{j=1}^d z_j \overline{w}_j$ we mean the standard inner product in $\mathbb{C}^d$. The reproducing kernel Hilbert space (RKHS) $\mathcal{H}(k_d)$ associated with k_d via Aronszajn's construction [7] is a natural multivariable analogue of the Hardy space H^2 of the unit disk and coincides with H^2 if $d = 1$.

For $\mathcal{Y}$ an auxiliary Hilbert space, we consider the tensor product Hilbert space $\mathcal{H}_{\mathcal{Y}}(k_d) := \mathcal{H}(k_d) \otimes \mathcal{Y}$ whose elements can be viewed as $\mathcal{Y}$-valued functions in $\mathcal{H}(k_d)$. Then $\mathcal{H}_{\mathcal{Y}}(k_d)$ can be characterized as follows:

$$\mathcal{H}_{\mathcal{Y}}(k_d) = \left\{ f(z) = \sum_{\mathbf{n}\in\mathbb{Z}^d_+} f_{\mathbf{n}} z^{\mathbf{n}} : \|f\|^2 = \sum_{\mathbf{n}\in\mathbb{Z}^d_+} \frac{\mathbf{n}!}{|\mathbf{n}|!} \cdot \|f_{\mathbf{n}}\|^2_{\mathcal{Y}} < \infty \right\}.$$

Here and in what follows, we use standard multivariable notations: for multi-integers $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d_+$ and points $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ we set

$$|\mathbf{n}| = n_1 + n_2 + \cdots + n_d, \qquad \mathbf{n}! = n_1!n_2!\dots n_d!, \qquad z^{\mathbf{n}} = z_1^{n_1} z_2^{n_2} \dots z_d^{n_d}.$$

By $\mathcal{M}_d(\mathcal{U},\mathcal{Y})$ we denote the space of all $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued analytic functions S on $\mathbb{B}^d$ such that the induced multiplication operator

$$M_S: \ f(z) \to S(z)\cdot f(z)$$

maps $\mathcal{H}_{\mathcal{U}}(k_d)$ into $\mathcal{H}_{\mathcal{Y}}(k_d)$. It follows by the closed graph theorem that for every $S \in \mathcal{M}_d(\mathcal{U},\mathcal{Y})$, the operator M_S is bounded. We shall pay particular attention to the unit ball of $\mathcal{M}_d(\mathcal{U},\mathcal{Y})$, denoted by

$$\mathcal{S}_d(\mathcal{U},\mathcal{Y}) = \{S \in \mathcal{M}_d(\mathcal{U},\mathcal{Y})\colon \|M_S\|_{\rm op} \le 1\}.$$

We refer to $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ as a generalized (d-variable) *Schur class* since $\mathcal{S}_1(\mathcal{U},\mathcal{Y})$ collapses to the classical Schur class. Characterizations of $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ in terms of realizations originate in [3, 15, 25]. The following is the analogue of Theorem 1.1 for this setting; the result with condition $(1'')$ eliminated appeared, e.g., in [15, 11].

Theorem 2.1. *Let S be an $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function defined on $\mathbb{B}^d$. The following are equivalent:*

$(1')$ *S belongs to $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$, i.e., the multiplication operator $M_S\colon f(z)\mapsto S(z)f(z)$ defines a contraction from $\mathcal{H}_{\mathcal{U}}(k_d)$ into $\mathcal{H}_{\mathcal{Y}}(k_d)$.*

$(1'')$ *$\|S(\mathbf{T})\| \le 1$ for any commutative row contraction $T=(T_1,\dots,T_d)\in\mathcal{L}(\mathcal{K})^d$, i.e., if S is given by $S(z)=\sum_{\mathbf{n}\in\mathbb{Z}^d_+} S_{\mathbf{n}} z^{\mathbf{n}}$ and if $(T_1,\dots,T_d)$ is any commuting d-tuple of bounded linear operators on a Hilbert space $\mathcal{K}$ such that the row matrix $\begin{bmatrix} T_1 & \cdots & T_d\end{bmatrix}$ defines a strict contraction operator from $\mathcal{K}^d$ to $\mathcal{K}$, then the operator $S(\mathbf{T})\in\mathcal{L}(\mathcal{U}\otimes\mathcal{K},\mathcal{Y}\otimes\mathcal{K})$ defined via the operator-norm limit of the series $S(T):=\sum_n S_n\otimes\mathbf{T}^{\mathbf{n}}$ has $\|S(T)\|\le 1$.*

(2) *The function $K_S\colon \mathbb{B}\times\mathbb{B}\to\mathcal{L}(\mathcal{Y})$ given by*

$$K_S(z,w) = \frac{I_{\mathcal{Y}} - S(z)S(w)^*}{1-\langle z,w\rangle}$$

is a positive kernel (see Proposition 1.2).

(3) *There exists a Hilbert space $\mathcal{H}$ and a unitary (or even coisometric or contractive) connecting operator (or colligation) $\mathbf{U}$ of the form*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D\end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D\end{bmatrix} : \begin{bmatrix}\mathcal{H}\\ \mathcal{U}\end{bmatrix} \to \begin{bmatrix}\mathcal{H}^d \\ \mathcal{Y}\end{bmatrix}$$

so that $S(z)$ can be realized in the form

$$\begin{aligned} S(z) &= D + C\left(I_{\mathcal{H}} - z_1A_1 - \cdots - z_dA_d\right)^{-1}(z_1B_1+\cdots+z_dB_d) \\ &= D + C(I-Z(z)A)^{-1}Z(z)B \end{aligned}$$

where we set

$$Z(z) = \begin{bmatrix} z_1 I_{\mathcal{H}} & \dots & z_d I_{\mathcal{H}}\end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d\end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_d\end{bmatrix}. \tag{2.1}$$

Remarks on the proof: (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1$''$) $\Longrightarrow$ (1$'$) follows in the same way as in the sketch of the proof of Theorem 1.1 above. For (1$'$) $\Longrightarrow$ (2), one can use the same reproducing kernel argument as the shortcut discussed in the proof Theorem 1.1 above. For (1$'$) $\Longrightarrow$ (1$''$), one can follow the corresponding argument sketched above for Theorem 1.1 but with the Sz.-Nagy dilation theorem replaced with the Drury dilation theorem (see [24]). The implication (2) $\Longrightarrow$ (3) follows exactly as in the classical case via the lurking isometry argument (see [15]). Note that (1$''$) $\Longrightarrow$ (2) also can be achieved directly by the GNS/HB argument in [23] specialized to the setting here, but this is not usually done since one has the alternative easier route (1$''$) $\Longrightarrow$ (1$'$) $\Longrightarrow$ (2). □

2.2. Free semigroup algebras

We now discuss the generalization of the Schur class associated with free semigroup algebras and models for row contractions (see [39, 40, 41, 42, 17]). We follow the formalism and notation as used in [17, 16].

Let $z = (z_1, \dots, z_d)$ and $w = (w_1, \dots, w_d)$ be two sets of noncommuting indeterminates. We let $\mathcal{F}_d$ denote the free semigroup generated by the d letters $\{1, \dots, d\}$. A generic element of $\mathcal{F}_d$ is a word w equal to a string of letters

$$\alpha = i_N \cdots i_1 \quad \text{where} \quad i_k \in \{1, \dots, d\} \text{ for } k = 1, \dots, N. \tag{2.2}$$

The product of two words is defined by the usual concatenation. The unit element of $\mathcal{F}_d$ is the *empty word* denoted by $\emptyset$. For α a word of the form (2.2), we let z^α denote the monomial in noncommuting indeterminates $z^\alpha = z_{i_N} \cdots z_{i_1}$ and we let $z^\emptyset = 1$. We extend this noncommutative functional calculus to a d-tuple of operators $\mathbf{A} = (A_1, \dots, A_d)$ on a Hilbert space $\mathcal{K}$:

$$\mathbf{A}^v = A_{i_N} \cdots A_{i_1} \quad \text{if} \quad v = i_N \cdots i_1 \in \mathcal{F}_d \setminus \{\emptyset\}; \quad \mathbf{A}^\emptyset = I_\mathcal{K}.$$

We will also have need of the *transpose operation* on $\mathcal{F}_d$:

$$\alpha^\top = i_1 \cdots i_N \quad \text{if} \quad \alpha = i_N \cdots i_1.$$

Given a coefficient Hilbert space $\mathcal{Y}$ we let $\mathcal{Y}\langle z \rangle$ denote the set of all polynomials in $z = (z_1, \dots, z_d)$ with coefficients in $\mathcal{Y}$:

$$\mathcal{Y}\langle z \rangle = \left\{ p(z) = \sum_{\alpha \in \mathcal{F}_d} p_\alpha z^\alpha : p_\alpha \in \mathcal{Y} \quad \text{and} \quad p_\alpha = 0 \text{ for all but finitely many } \alpha \right\},$$

while $\mathcal{Y}\langle\langle z \rangle\rangle$ denotes the set of all formal power series in the indeterminates z with coefficients in $\mathcal{Y}$:

$$\mathcal{Y}\langle\langle z \rangle\rangle = \left\{ f(z) = \sum_{\alpha \in \mathcal{F}_d} f_\alpha z^\alpha : f_\alpha \in \mathcal{Y} \right\}.$$

Note that vectors in $\mathcal{Y}$ can be considered as Hilbert space operators between $\mathbb{C}$ and $\mathcal{Y}$. More generally, if $\mathcal{U}$ and $\mathcal{Y}$ are two Hilbert spaces, we let $\mathcal{L}(\mathcal{U}, \mathcal{Y})\langle z \rangle$ and $\mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ denote the space of polynomials (respectively, formal power series) in the noncommuting indeterminates $z = (z_1, \dots, z_d)$ with coefficients in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$.

Given $S = \sum_{\alpha \in \mathcal{F}_d} s_\alpha z^\alpha \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ and $f = \sum_{\beta \in \mathcal{F}_d} f_\beta z^\beta \in \mathcal{U}\langle\langle z \rangle\rangle$, the product $S(z) \cdot f(z) \in \mathcal{Y}\langle\langle z \rangle\rangle$ is defined as an element of $\mathcal{Y}\langle\langle z \rangle\rangle$ via the noncommutative convolution:

$$S(z) \cdot f(z) = \sum_{\alpha, \beta \in \mathcal{F}_d} s_\alpha f_\beta z^{\alpha\beta} = \sum_{v \in \mathcal{F}_d} \left(\sum_{\alpha, \beta \in \mathcal{F}_d \colon \alpha \cdot \beta = v} s_\alpha f_\beta \right) z^v. \tag{2.3}$$

Note that the coefficient of z^v in (2.3) is well defined since any given word $v \in \mathcal{F}_d$ can be decomposed as a product $v = \alpha \cdot \beta$ in only finitely many distinct ways.

In general, given a coefficient Hilbert space $\mathcal{C}$, we use the $\mathcal{C}$ inner product to generate a pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{C} \times \mathcal{C}\langle\langle w \rangle\rangle} \colon \mathcal{C} \times \mathcal{C}\langle\langle w \rangle\rangle \to \mathbb{C}\langle\langle w \rangle\rangle$$

via

$$\left\langle c, \sum_{\beta \in \mathcal{F}_d} f_\beta w^\beta \right\rangle_{\mathcal{C} \times \mathcal{C}\langle\langle w \rangle\rangle} = \sum_{\beta \in \mathcal{F}_d} \langle c, f_\beta \rangle_{\mathcal{C}} w^{\beta^\top} \in \mathbb{C}\langle\langle w \rangle\rangle.$$

Similarly we can consider $\left\langle \sum_{\alpha \in \mathcal{F}_d} f_\alpha w^\alpha, c \right\rangle_{\mathcal{C}\langle\langle w \rangle\rangle \times \mathcal{C}}$ or the more general pairing

$$\left\langle \sum_{\alpha \in \mathcal{F}_d} f_\alpha w'^\alpha, \sum_{\beta \in \mathcal{F}_d} g_\beta w^\beta \right\rangle_{\mathcal{C}\langle\langle w' \rangle\rangle \times \mathcal{C}\langle\langle w \rangle\rangle} = \sum_{\alpha, \beta \in \mathcal{F}_d} \langle f_\alpha, g_\beta \rangle_{\mathcal{C}} w^{\beta^\top} w'^\alpha.$$

Suppose that $\mathcal{H}$ is a Hilbert space whose elements are formal power series in $\mathcal{Y}\langle\langle z \rangle\rangle$ and that $K(z, w) = \sum_{\alpha, \beta \in \mathcal{F}_d} K_{\alpha,\beta} z^\alpha w^{\beta^\top}$ is a formal power series in the two sets of d noncommuting indeterminates $z = (z_1, \dots, z_d)$ and $w = (w_1, \dots, w_d)$. We say that $\mathcal{H}$ is a NFRKHS (*noncommutative formal reproducing kernel Hilbert space*) if for each $\alpha \in \mathcal{F}_d$, the linear operator $\Phi_\alpha \colon \mathcal{H} \to \mathcal{Y}$ defined by $f(z) = \sum_{\beta \in \mathcal{F}_d} f_\beta z^\beta \mapsto f_\alpha$ is continuous. In this case there must be a formal power series $k_\alpha(z) \in \mathcal{L}(\mathcal{Y})\langle\langle z \rangle\rangle$ so that $k_\alpha(\cdot) y \in \mathcal{H}$ for each $\alpha \in \mathcal{F}_d$ and $y \in \mathcal{Y}$ and

$$\langle f, k_\alpha y \rangle_{\mathcal{H}} = \langle f_\alpha, y \rangle_{\mathcal{Y}}.$$

If we set $K(z, w) = \sum_{\beta \in \mathcal{F}_d} k_\beta(z) w^{\beta^\top}$, then we have the reproducing property

$$\langle f, K(\cdot, w) y \rangle_{\mathcal{H} \times \mathcal{H}\langle\langle w \rangle\rangle} = \langle f(w), y \rangle_{\mathcal{Y}\langle\langle w \rangle\rangle \times \mathcal{Y}}.$$

In this case we say that $K(z, w)$ *is the reproducing kernel for the NFRKHS* $\mathcal{H}$. As explained in detail in [16], we have the following equivalent characterizations for such kernels which parallel the statements of Proposition 1.2 for the classical case.

Proposition 2.2. *Let $K(z, w) \in \mathcal{L}(\mathcal{Y})\langle\langle z, w \rangle\rangle$ be a formal power series in two sets of noncommuting indeterminates with coefficients $K_{\alpha,\beta}$ equal to bounded operators on the Hilbert space $\mathcal{Y}$. Then the following conditions are equivalent:*

1. *For all finitely supported $\mathcal{Y}$-valued functions $\alpha \mapsto y_\alpha$ it holds that*

$$\sum_{\alpha, \alpha' \in \mathcal{F}_d} \langle K_{\alpha,\alpha'} y_{\alpha'}, y_\alpha \rangle \geq 0,$$

i.e., the function from $\mathcal{F}_d \times \mathcal{F}_d$ *to* $\mathcal{L}(\mathcal{Y})$ *given by* $(\alpha, \beta) \mapsto K_{\alpha,\beta}$ *is a positive kernel in the classical sense of Proposition* 1.2*.*

2. $K(z,w)$ *has a factorization*

$$K(z,w) = H(z)H(w)^*$$

for some $H \in \mathcal{L}(\mathcal{H}, \mathcal{Y})\langle\langle z \rangle\rangle$ *where* $\mathcal{H}$ *is some auxiliary Hilbert space. Here*

$$H(w)^* = \sum_{\beta \in \mathcal{F}_d} H_\beta^* w^{\beta^\top} = \sum_{\beta \in \mathcal{F}_d} H_{\beta^\top}^* w^\beta \quad \textit{if} \quad H(z) = \sum_{\alpha \in \mathcal{F}_d} H_\alpha z^\alpha.$$

3. $K(z,w)$ *is the reproducing kernel for a NFRKHS* $\mathcal{H}(K)$*, i.e., for each* $\beta \in \mathcal{F}_d$ *and* $y \in \mathcal{Y}$ *the formal power series* $k_\beta y$ *given by* $k_\beta y(z) := \sum_{\alpha \in \mathcal{F}_d} K_{\alpha,\beta} y z^\alpha$ *is in* $\mathcal{H}(K)$ *and has the reproducing property*

$$\left\langle f, \sum_{\beta \in \mathcal{F}_d} k_\beta y w^\beta \right\rangle_{\mathcal{H}(K) \times \mathcal{H}(K)\langle\langle w \rangle\rangle} = \langle f(w), y \rangle_{\mathcal{Y}\langle\langle w \rangle\rangle \times \mathcal{Y}} \textit{ for every } f \in \mathcal{H}(K).$$

A natural analogue of the vector-valued Hardy space over the unit disk (see, e.g., [39]) is the Fock space with coefficients in $\mathcal{Y}$ which we denote here by $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ and express the elements in power series form:

$$H^2_{\mathcal{Y}}(\mathcal{F}_d) = \left\{ f(z) = \sum_{\alpha \in \mathcal{F}_d} f_\alpha z^\alpha \colon f_\alpha \in \mathcal{Y}, \sum_{\alpha \in \mathcal{F}_d} \|f_\alpha\|^2 < \infty \right\}. \tag{2.4}$$

When $\mathcal{Y} = \mathbb{C}$ we write simply $H^2(\mathcal{F}_d)$. As explained in [16], $H^2(\mathcal{F}_d)$ is a NFRKHS with reproducing kernel equal the following noncommutative analogue of the classical Szegö kernel:

$$k_{\rm Sz,\ nc}(z,w) = \sum_{\alpha \in \mathcal{F}_d} z^\alpha w^{\alpha^\top}. \tag{2.5}$$

Thus we have in general $H^2_{\mathcal{Y}}(\mathcal{F}_d) = \mathcal{H}(k_{\rm Sz} \otimes I_{\mathcal{Y}})$. We abuse notation and let S_j denote the shift operator

$$S_j \colon f(z) = \sum_{v \in \mathcal{F}_d} f_v z^v \mapsto f(z) \cdot z_j = \sum_{v \in \mathcal{F}_d} f_v z^{v \cdot j} \text{ for } j = 1, \dots, d$$

on $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ for any auxiliary space $\mathcal{Y}$. The adjoint of S_j on $H^2_{\mathcal{Y}}(\mathcal{F}_d)$ is then given by

$$S_j^* \colon \sum_{v \in \mathcal{F}_d} f_v z^v \mapsto \sum_{v \in \mathcal{F}_d} f_{v \cdot j} z^v \quad \text{for} \quad j = 1, \dots, d.$$

We let $\mathcal{M}_{nc,d}(\mathcal{U}, \mathcal{Y})$ denote the set of formal power series $S(z) = \sum_{\alpha \in \mathcal{F}_d} s_\alpha z^\alpha$ with coefficients $s_\alpha \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ such that the associated multiplication operator $M_S \colon f(z) \mapsto S(z) \cdot f(z)$ (see (2.3)) defines a bounded operator from $H^2_{\mathcal{U}}(\mathcal{F}_d)$ to $H^2_{\mathcal{Y}}(\mathcal{F}_d)$. The noncommutative Schur class $\mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ is defined to consist of such multipliers S for which M_S has operator norm at most 1:

$$\mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y}) = \{ S \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle \colon M_S \colon H^2_{\mathcal{U}}(\mathcal{F}_d) \to H^2_{\mathcal{Y}}(\mathcal{F}_d) \text{ with } \|M_S\|_{op} \le 1 \}.$$

The following is the noncommutative analogue of Theorem 1.1 for this setting. We refer to [39, 40] for details.

Theorem 2.3. *Let $S(z) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ be a formal power series in $z = (z_1, \ldots, z_d)$ with coefficients in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. Then the following are equivalent:*

(1′) *$S \in \mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$, i.e., $M_S \colon \mathcal{U}\langle z \rangle \to \mathcal{Y}\langle\langle z \rangle\rangle$ given by $M_S \colon p(z) \to S(z)p(z)$ extends to define a contraction operator from $H^2_{\mathcal{U}}(\mathcal{F}_d)$ into $H^2_{\mathcal{Y}}(\mathcal{F}_d)$.*

(1″) *For each strict row contraction $(T_1, \ldots, T_d)$, i.e., a d-tuple $(T_1, \ldots, T_d)$ of operators on a Hilbert space $\mathcal{K}$ (commutative or not) such that the row matrix $\begin{bmatrix} T_1 & \cdots & T_d \end{bmatrix}$ defines a strict contraction operator from $\mathcal{K}^d$ to $\mathcal{K}$, we have*

$$\|S(T)\| \le 1,$$

where

$$S(T) = \sum_{\alpha \in \mathcal{F}_d} s_\alpha \otimes T^\alpha \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K}), \mathcal{Y} \otimes \mathcal{K}) \quad \text{if} \quad S(z) = \sum_{\alpha \in \mathcal{F}_d} s_\alpha z^\alpha$$

and where we set

$$T^\alpha = T_{i_N} \cdots T_{i_1} \quad \text{if} \quad \alpha = i_N \cdots i_1 \in \mathcal{F}_d.$$

(2) *The formal power series given by*

$$K_S(z, w) := k_{Sz,\ nc}(z, w) - S(z) k_{Sz,\ nc}(z, w) S(w)^*$$

is a noncommutative positive kernel (see Proposition 2.2).

(3) *There exists a Hilbert space $\mathcal{H}$ and a unitary connection operator $\mathbf{U}$ of the form*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \vdots \\ \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

so that $S(z)$ can be realized as a formal power series in the form

$$S(z) = D + \sum_{j=1}^{d} \sum_{v \in \mathcal{F}_d} C A^v B_j z^v \cdot z_j = D + C(I - Z(z)A)^{-1} Z(z) B$$

where $Z(z)$, A and B are as in (2.1) but where now $z_1, \ldots, z_d$ are noncommuting indeterminates rather than commuting variables.

Sketch of the proof of Theorem 2.3: The proof of (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1″) $\Longrightarrow$ (1′) $\Longrightarrow$ (1) formally goes through in the same was as the classical case. Let us just note that (1″) $\Longrightarrow$ (1′) involves viewing $M_S \colon H^2_{\mathcal{U}}(\mathcal{F}_d) \to H^2_{\mathcal{Y}}(k_d)$ as $M_S = S(\mathbf{S})$ where $\mathbf{S} = (S_1, \ldots, S_d)$ are the left creation operators of multiplication by z_j on the left on the Fock space $H^2(\mathcal{F}_d)$. From the assumption (1″), we know that $\|S(r\mathbf{S})\| \le 1$ for each $r < 1$ and hence $\|M_S\| = \lim_{r \uparrow 1} \|S(r\mathbf{S})\| \le 1$ as well.

We discuss the harder direction (1′) $\Longrightarrow$ (1″) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

(1′) $\Longrightarrow$ (1″): One can follow the proof of (1′) $\Longrightarrow$ (1″) for the classical case but substitute the Popescu dilation theorem for row contractions (see [37]) for the Sz.-Nagy dilation theorem for a single contraction operator.

(1″) $\Longrightarrow$ (2): This implication again can be done via an appropriate version of the GNS/HB argument; see [12] for a slightly more general version and [23] for an even more general version.

Alternatively, one can follow the route **(1″) $\Longrightarrow$ (1′) $\Longrightarrow$ (2):** As we have already discussed (1″) $\Longrightarrow$ (1′), it suffices to discuss (1′) $\Longrightarrow$ (2). This can be done by an adaptation of the argument for the classical case to the present setting of formal, noncommutative reproducing kernel Hilbert spaces – see [16, Theorem 3.15].

(2) $\Longrightarrow$ (3): The lurking isometry argument works in this context as well – see [16, Theorem 3.16]. □

3. Reproducing kernel $(\mathcal{A}, \mathcal{B})$-correspondences

The notion of a vector-valued reproducing kernel Hilbert space based on an operator-valued positive kernel has been a standard tool in operator theory as well as in other applications for some time now. Recently, Barreto, Bhat, Liebscher and Skeide [18] introduced a finer notion of positive kernel (*completely positive kernel*) and gave several equivalent characterizations, but did not develop the connections with reproducing kernel Hilbert spaces. The purpose of this section is to fill in this gap, as it is the natural tool for the discussion to follow.

Let $\mathcal{B}$ be a C^*-algebra and E a linear space. For some of the discussion to follow, it will be convenient to assume that $\mathcal{B}$ has a unit. However, any C^*-algebra has an *approximate identity* (see [20, Theorem I.4.8]); by making use of such an approximate identity, most arguments using a unit element $1_\mathcal{B}$ can be adapted to an approximation argument yielding the desired result for the general case where $\mathcal{B}$ is not assumed to possess a unit. In the sequel we usually leave the details of this adaptation to the reader.

We say that E is a (*right*) *pre-Hilbert C^*-module* over $\mathcal{B}$ if E is a right module over $\mathcal{B}$ and is endowed with a $\mathcal{B}$-valued inner product $\langle \cdot, \cdot \rangle_E$ satisfying the following axioms for any $\lambda, \mu \in \mathbb{C}$, $e, f, g \in E$ and $b \in \mathcal{B}$:

1. $\langle \lambda e + \mu f, g \rangle_E = \lambda \langle e, g \rangle_E + \mu \langle f, g \rangle_E$;
2. $\langle e \cdot b, f \rangle_E = \langle e, f \rangle_E b$;
3. $\langle e, f \rangle_E^* = \langle f, e \rangle_E$;
4. $\langle e, e \rangle_E \geq 0$ (as an element of $\mathcal{B}$);
5. $\langle e, e \rangle_E = 0$ implies that $e = 0$.

We also impose that $(\lambda e) \cdot b = e \cdot (\lambda b)$ for all $e \in E$, $b \in \mathcal{B}$ and $\lambda \in \mathbb{C}$. Note that if $\mathcal{B}$ has a unit, this last condition is automatic from the axioms for the identification $\lambda \mapsto \lambda \cdot 1_\mathcal{B}$ and the axioms for E being a module over $\mathcal{B}$. (Unlike some other authors, we take the $\mathcal{B}$-valued inner product to be linear in the first variable and conjugate-linear in the second variable as is usually done in the Hilbert-space setting ($\mathcal{B} = \mathbb{C}$)

rather than the reverse.) Note that it then follows that

$$\langle e, f \cdot b\rangle_E = b^* \langle e, f\rangle_E.$$

When the inner product is clear, we drop the subscript E and write simply $\langle e, f\rangle$ for the $\mathcal{B}$-valued inner product. If E is a pre-Hilbert module over $\mathcal{B}$, then E is a normed linear space with norm given by

$$\|e\|_E = \|\langle e, e\rangle^{1/2}\|_{\mathcal{B}}. \tag{3.1}$$

Here $\| \ \|_{\mathcal{B}}$ denotes the norm associated with the C^*-algebra $\mathcal{B}$. One can always complete E to a Banach space in the norm (3.1) to get what we shall call a *Hilbert C^*-module over $\mathcal{B}$*. Moreover, E has additional structure, namely E carries the structure of an *operator space*, i.e., E is the upper right corner of a subalgebra of operators acting on a Hilbert space with a representation as 2×2-block operator matrices (the *linking algebra*) – see [31] or [43].

Given two Hilbert C^*-modules E and F over the same C^*-algebra $\mathcal{B}$, it is natural to consider the space $\mathcal{L}(E, F)$ of bounded linear operators $T\colon E \to F$ between the Banach spaces E and F. Unlike the Hilbert space case, for a linear map T from E to F it may or may not happen that there is an *adjoint operator* $T^* \in \mathcal{L}(F, E)$ so that

$$\langle Te, f\rangle_F = \langle e, T^* f\rangle_E \text{ for all } e \in E \text{ and } f \in F.$$

In case there exists an operator $T^* \in \mathcal{L}(F, E)$ with this property we say that T is *adjointable* and we denote the set of all adjointable linear operators between E and F as $\mathcal{L}^a(E, F)$ (with the usual abbreviation $\mathcal{L}^a(E)$ in case $E = F$). When the mapping $T\colon E \to F$ is adjointable in this sense, necessarily $T \in \mathcal{L}(E, F)$ with the additional property that T is a $\mathcal{B}$-module map:

$$T(e \cdot b) = T(e) \cdot b \text{ for all } e \in E \quad \text{and} \quad b \in \mathcal{B}. \tag{3.2}$$

However, this additional property (3.2) alone is not sufficient for admission of T in the class $\mathcal{L}^a(E, F)$ of adjointable maps (see [43, Example 2.19]).

Following [31, 33] (see also the books [29, 43] for more comprehensive treatments), we now introduce the notion of an $(\mathcal{A}, \mathcal{B})$-correspondence. If E is a right Hilbert C^*-module over $\mathcal{B}$ and $\mathcal{A}$ is another C^*-algebra, we say that E is a $(\mathcal{A}, \mathcal{B})$-*correspondence* if E is also a left module over $\mathcal{A}$ which makes E an $(\mathcal{A}, \mathcal{B})$-bimodule:

$$(a \cdot e) \cdot b = a \cdot (e \cdot b) \text{ for all } a \in \mathcal{A},\ e \in E \quad \text{and} \quad b \in \mathcal{B}$$

with the additional compatibility condition

$$\langle a \cdot e,\ f\rangle_E = \langle e,\ a^* \cdot f\rangle_E. \tag{3.3}$$

The compatibility condition in (3.3) is equivalent to requiring that each of the left multiplication operators $\varphi(a)\colon e \mapsto a \cdot e$ on E is a bounded linear operator on E for each $a \in \mathcal{A}$ and φ is a C^*-homomorphism from $\mathcal{A}$ into the C^*-algebra $\mathcal{L}^a(E)$ of bounded adjointable operators on E: thus $\varphi(a)$ is adjointable for each $a \in \mathcal{A}$ with $\varphi(a)^* = \varphi(a^*)$. We shall occasionally write $\varphi(a)e$ rather than $a \cdot e$.

Note the lack of symmetry in the roles of $\mathcal{A}$ and $\mathcal{B}$: the identities $\langle e \cdot b, f\rangle = \langle e, f\rangle b$ together with $\langle e, f \cdot b^*\rangle = b \cdot \langle e, f\rangle$ preclude the validity in general of the identity $\langle e \cdot b, f\rangle = \langle e, f \cdot b^*\rangle$ (the would-be $\mathcal{B}$ analogue of (3.3)) unless $\mathcal{B}$ is commutative.

If both $\mathcal{A}$ and $\mathcal{B}$ have units, we also demand that the scalar multiplication on E is compatible with both the identification $\lambda \mapsto \lambda 1_{\mathcal{A}}$ of $\mathbb{C}$ as a subalgebra of $\mathcal{A}$ and the identification $\lambda \mapsto \lambda 1_{\mathcal{B}}$ of $\mathbb{C}$ as a subalgebra of $\mathcal{B}$. This is consistent with demanding the additional axioms

$$(\lambda a) \cdot e = a \cdot (\lambda e), \qquad (\lambda e) \cdot b = e \cdot (\lambda b)$$

for the general case.

The classical case is the one where E is a Hilbert space $\mathcal{E}$, $\mathcal{B} = \mathbb{C}$ and $\mathcal{A} = \mathcal{L}(\mathcal{E})$ with the operations given by

$$\begin{aligned}
&a \cdot e = ae \text{ (the operator } a \text{ acting on the vector } e)\\
&e \cdot b = be \text{ (scalar multiplication in } \mathcal{E}),\\
&\langle e, f\rangle \text{ (the } \mathcal{E} \text{ Hilbert-space inner product).}
\end{aligned}$$

Another easy example is to take $E = \mathcal{A} = \mathcal{B}$ all equal to a C^*-algebra with

$$a \cdot e = ae, \qquad e \cdot b = eb, \qquad \langle e, f\rangle_E = f^* e.$$

We encourage the reader to peruse Section 6 for a variety of additional examples and references for more complete details.

We will have need of various constructions for making new correspondences out of given correspondences. We give formal definitions as follows.

Definition 3.1.

1. **Direct sum:** Let E and F be two $(\mathcal{A}, \mathcal{B})$-correspondences. Then the direct-sum correspondence $E \oplus F$ is defined to be the direct sum vector space $E \oplus F$ together with the diagonal left-$\mathcal{A}$ action and right-$\mathcal{B}$ action and the direct-sum $\mathcal{B}$-valued inner product:
$$\begin{aligned}
&a \cdot (e \oplus f) = (a \cdot e) \oplus (a \cdot f), \qquad (e \oplus f) \cdot b = (e \cdot b) \oplus (f \cdot b),\\
&\langle e \oplus f, e' \oplus f'\rangle_{E \oplus F} = \langle e, e'\rangle_E + \langle f, f'\rangle_F.
\end{aligned}$$
2. **Tensor product:** Suppose that we are given three C^*-algebras $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ together with an $(\mathcal{A}, \mathcal{B})$-correspondence E and a $(\mathcal{B}, \mathcal{C})$-correspondence F. Then we define the tensor product correspondence $E \otimes_{\mathcal{B}} F$ (sometimes abbreviated to $E \otimes F$) to be the completion of the linear span of all tensors $e \otimes f$ (with $e \in E$ and $f \in F$) subject to the identification
$$(e \cdot b) \otimes f = e \otimes (b \cdot f), \tag{3.4}$$
with left $\mathcal{A}$-action given by
$$a \cdot (e \otimes f) = (a \cdot e) \otimes f,$$

with right $\mathcal{C}$-action given by

$$(e \otimes f) \cdot c = e \otimes (f \cdot c),$$

and with $\mathcal{C}$-valued inner product $\langle \cdot, \cdot \rangle_{E \otimes F}$ given by

$$\langle e \otimes f,\, e' \otimes f' \rangle_{E \otimes F} = \langle \langle e, e' \rangle_E \cdot f,\, f' \rangle_F \,.$$

It is a straightforward exercise to verify that the balanced tensor-product construction is well defined. For example the computation

$$\begin{aligned}\langle (e \cdot b) \otimes f,\, (e' \cdot b') \otimes f' \rangle &= \langle b'^* \cdot \langle e, e' \rangle \cdot b \cdot f, f' \rangle \\ &= \langle \langle e, e' \rangle \cdot b \cdot f,\, b' \cdot f' \rangle \\ &= \langle e \otimes (b \cdot f),\, e' \otimes (b' \cdot f') \rangle\end{aligned}$$

shows that the $E \otimes F$-inner product is well defined.

Remark 3.2. Bounded linear operators between direct sum correspondences admit operator matrix decompositions in precisely the same way as in the Hilbert space case ($\mathcal{B} = \mathbb{C}$), while adjointability of such an operator corresponds to the operators in the decomposition being adjointable. For bounded linear operators between tensor product correspondences the situation is slightly more complicated. We give an example how operators can be constructed. Let E and E' be $(\mathcal{A}, \mathcal{B})$-correspondences and F and F' $(\mathcal{B}, \mathcal{C})$-correspondences, for C^*-algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Furthermore, let $X \in \mathcal{L}(E, E')$ and $Y \in \mathcal{L}(F, F')$ be $\mathcal{B}$-module maps. Then we write $X \otimes Y$ for the operator in $\mathcal{L}(E \otimes_{\mathcal{B}} F, E' \otimes_{\mathcal{B}} F')$ which is determined by

$$X \otimes Y(e \otimes f) = (Xe) \otimes (Yf) \text{ for each } e \otimes f \in E \otimes_{\mathcal{B}} F. \tag{3.5}$$

The $\mathcal{B}$-module map properties are needed to guarantee that for each $e \otimes f \in E \otimes_{\mathcal{B}} F$ and all $b \in \mathcal{B}$ we have

$$\begin{aligned}X \otimes Y(eb \otimes f) &= (X(eb)) \otimes (Yf) = (Xe)b \otimes (Yf) = (Xe) \otimes b(Yf) \\ &= (Xe) \otimes (Y(bf)) = X \otimes Y(e \otimes bf).\end{aligned}$$

Thus the balancing in the tensor product (see (3.4)) is respected by the operator $X \otimes Y$. Moreover, $X \otimes Y$ is adjointable in case X and Y are adjointable operators, with $(X \otimes Y)^* = X^* \otimes Y^*$. Indeed, this is the case since for $f \otimes g \in E \otimes F$ and $e' \otimes f' \in E' \otimes F'$ we have

$$\begin{aligned}\langle (X \otimes Y)(e \otimes f),\, e' \otimes f' \rangle_{E' \otimes F'} &= \langle Xe \otimes Yf,\, e' \otimes f' \rangle_{E' \otimes F'} \\ &= \langle \langle Xe,\, e' \rangle_{E'} Yf,\, f' \rangle_{F'} \\ &= \langle Y \langle Xe,\, e' \rangle_{E'} f,\, f' \rangle_{F'} \\ &= \langle \langle e,\, X^* e' \rangle_E f,\, Y^* f' \rangle_F \\ &= \langle e \otimes f,\, X^* e' \otimes Y^* f' \rangle_{E \otimes F} \\ &= \langle e \otimes f,\, (X^* \otimes Y^*) e' \otimes f' \rangle_{E \otimes F}.\end{aligned}$$

In particular, the left action on $E \otimes F$ can now be written as $a \mapsto \varphi(a) \otimes I_F \in \mathcal{L}^a(E \otimes F, E \otimes F)$, where $I_F \in \mathcal{L}^a(F, F)$ is the identity operator on F. We will have occasions to use operators constructed in this way in the sequel.

We now introduce the notion of *reproducing kernel* $(\mathcal{A}, \mathcal{B})$*-correspondence*.

Definition 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. By an $(\mathcal{A}, \mathcal{B})$*-reproducing kernel correspondence* on a set Ω, we mean an $(\mathcal{A}, \mathcal{B})$-correspondence E whose elements are $\mathcal{B}$-valued functions $f\colon (\omega, a) \mapsto f(\omega, a) \in \mathcal{B}$ on $\Omega \times \mathcal{A}$, which is a vector space with respect to the usual pointwise vector space operations and such that for each $\omega \in \Omega$ there is a *kernel element* $k_\omega \in E$ with

$$f(\omega, a) = \langle a \cdot f,\, k_\omega \rangle_E. \tag{3.6}$$

When this is the case we say that the function $\mathbb{K}\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})$ given by

$$\mathbb{K}(\omega, \omega')[a] = k_{\omega'}(\omega, a) \tag{3.7}$$

is the *reproducing kernel* for the reproducing kernel correspondence E.

From the inner product characterization in (3.6) of the point evaluation for elements in an $(\mathcal{A}, \mathcal{B})$-reproducing kernel correspondence E on Ω one easily deduces that the left $\mathcal{A}$-action and the right $\mathcal{B}$-action are given by

$$(a \cdot f)(\omega', a') = f(\omega', a'a) \quad \text{and} \quad (f \cdot b)(\omega', a') = f(\omega', a')b. \tag{3.8}$$

It is implicit in Definition 3.3 that the map $a \mapsto k_{\omega'}(\omega, a) \in \mathcal{B}$ is linear in $a \in \mathcal{A}$ for each $\omega, \omega' \in \Omega$. In fact the mapping from $\mathcal{A}$ to $\mathcal{B}$ given by $a \mapsto f(\omega, a)$ is $\mathcal{A}$-linear for each fixed $f \in E$ and $\omega \in \Omega$. If $\mathcal{A}$ has a unit $1_\mathcal{A}$, this follows from the general identity $f(\omega, a) = (a \cdot f)(\omega, 1_\mathcal{A})$ (a consequence of (3.8) together with the linearity of the point-evaluation map $f \mapsto f(\omega, 1_\mathcal{A})$ from E to $\mathcal{B}$ for each fixed $\omega \in \Omega$ which in turn is an easy consequence of (3.6)). The general case follows by adapting this argument to the setting where one has only an approximate identity. Note also that we recover the element $k_{\omega'}$ from $\mathbb{K}$ by using formula (3.7) to define $k_{\omega'}$ as a function of (ω, a) for each $\omega' \in \Omega$.

The next proposition gives some elementary observations concerning the structure of reproducing kernel correspondences.

Proposition 3.4. *If E is a reproducing kernel $(\mathcal{A}, \mathcal{B})$-correspondence with kernel elements k_ω for $\omega \in \Omega$, then the bounded evaluation map $e_{\omega,a}$ from E to $\mathcal{B}$ given by $e_{\omega,a}\colon f \mapsto f(\omega, a)$ is adjointable for each fixed $(\omega, a) \in \Omega \times \mathcal{A}$ and we have*

$$a^* k_\omega b = e_{\omega,a}^* b \text{ for each } \omega \in \Omega,\ a \in \mathcal{A}, \text{ and } b \in \mathcal{B}. \tag{3.9}$$

Conversely, suppose that E is an $(\mathcal{A}, \mathcal{B})$-correspondence of $\mathcal{B}$-valued functions on the set $\Omega \times \mathcal{A}$ satisfying (3.8) *and such that the evaluation map*

$$e_{\omega,a}\colon f \mapsto f(\omega, a)$$

is a bounded and adjointable map from E to $\mathcal{B}$ for each $\omega \in \Omega$ and $a \in \mathcal{A}$. Then E is a reproducing kernel $(\mathcal{A}, \mathcal{B})$-correspondence with reproducing kernel elements determined by (3.9)*.*

Moreover, in either case, for each fixed pair (ω, a) the point-evaluation map $e_{\omega,a}\colon E \to \mathcal{B}$ is a $\mathcal{B}$-module map:

$$(f \cdot b)(\omega, a) = f(\omega, a)b \text{ for all } b \in \mathcal{B}.$$

Proof. Suppose E is a reproducing kernel $(\mathcal{A},\mathcal{B})$-correspondence with kernel elements k_ω for $\omega \in \Omega$. If $e_{\omega,a}$ denotes the evaluation map from E to $\mathcal{B}$ given by $e_{\omega,a}\colon f \mapsto f(\omega,a)$, we have

$$\langle e_{\omega,a}f, b\rangle_{\mathcal{B}} = b^* f(\omega,a) = b^*\langle a\cdot f, k_\omega\rangle_E = \langle f, a^* k_\omega \cdot b\rangle.$$

So $e_{\omega,a}$ is adjointable with $e^*_{\omega,a}b = a^* k_\omega b$ for any $b \in \mathcal{B}$.

On the other hand, if the evaluation map

$$e_{\omega,a}\colon f \mapsto f(\omega,a)$$

is a bounded and adjointable map from E to $\mathcal{B}$ for each $\omega \in \Omega$ and $a \in \mathcal{A}$, then there exists an $e^*_{\omega,a}$ so that

$$b^*(e_{\omega,a}f) = \langle e_{\omega,a}f, b\rangle_{\mathcal{B}} = \langle f, e^*_{\omega,a}b\rangle_E.$$

If $\mathcal{A}$ and $\mathcal{B}$ have identities $1_\mathcal{A}$ and $1_\mathcal{B}$ respectively, we set $k_\omega = e^*_{\omega,1_\mathcal{A}}(1_\mathcal{B})$. Using the first identity in (3.8) it follows from a computation similar to that above, that $a^*k_\omega = e^*_{\omega,a}(1_\mathcal{B})$. We readily see that

$$f(\omega,a) = e_{\omega,a}f = \langle f, a^*k_\omega\rangle_E = \langle a\cdot f, k_\omega\rangle_E.$$

If $\mathcal{A}$ and/or $\mathcal{B}$ does not have a unit, one can do an approximate version of the above argument using an approximate identity for $\mathcal{A}$ and/or $\mathcal{B}$. In any case, it follows that E is a reproducing kernel $(\mathcal{A},\mathcal{B})$-correspondence with reproducing kernel elements determined by (3.9).

The last part follows from the definition of the right $\mathcal{B}$-action given by (3.8). □

Given a reproducing kernel $(\mathcal{A},\mathcal{B})$-correspondence as in Definition 3.3, one can show that the associated reproducing kernel function $\mathbb{K}\colon \Omega\times\Omega \to \mathcal{L}(\mathcal{A},\mathcal{B})$ defined by (3.7) is a completely positive kernel in the sense of [18], i.e., the function

$$((\omega,a),(\omega',a')) \to \mathbb{K}(\omega,\omega')[a^*a']$$

is a positive kernel in the classical sense of Aronszajn [7] (extended to the C^*-algebra-valued case), that is, $\sum_{i,j=1}^N b_i^*\mathbb{K}(\omega_i,\omega_j)[a_i^*a_j]b_j$ *is a positive element of* $\mathcal{B}$ *for each choice of finitely many* $(\omega_1,a_1),\dots,(\omega_N,a_N)$ *in* $\Omega\times\mathcal{A}$ *and* $b_1,\dots,b_N$ *in* $\mathcal{B}$. In fact, by the axioms of an $(\mathcal{A},\mathcal{B})$-correspondence combined with the reproducing property of the kernel elements k_ω, we have

$$\begin{aligned}\sum_{i,j=1}^N b_i^*\mathbb{K}(\omega_i,\omega_j)[{a_i}^*a_j]b_j &= \sum_{i,j=1}^N b_i^*\langle {a_i}^*a_jk_{\omega_j}, k_{\omega_i}\rangle_E b_j\\ &= \sum_{i,j=1}^N \langle a_jk_{\omega_j}b_j, a_ik_{\omega_i}b_i\rangle_E\\ &= \left\langle \sum_{j=1}^N a_jk_{\omega_j}b_j, \sum_{i=1}^N a_ik_{\omega_i}b_i\right\rangle_E\\ &\geq 0.\end{aligned}$$

Actually, we have the following equivalent statements.

Theorem 3.5. *Given a function* $\mathbb{K}\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})$*, the following are equivalent:*

1. $\mathbb{K}$ *is a completely positive kernel in the sense that the function from* $(\Omega \times \mathcal{A}) \times (\Omega \times \mathcal{A}) \to \mathcal{L}(\mathcal{A}, \mathcal{B})$ *given by*
$$((\omega, a), (\omega', a')) \mapsto \mathbb{K}(\omega', \omega)[a^* a']$$
is a positive kernel in the sense of Aronszajn: for all $(\omega_1, a_1), \ldots, (\omega_N, a_N)$ *in* $\Omega \times \mathcal{A}$ *and* $b_1, \ldots, b_N$ *in* $\mathcal{B}$ *we have*
$$\sum_{i,j=1}^{N} b_i^* \mathbb{K}(\omega_i, \omega_j)[{a_i}^* a_j] b_j \geq 0 \text{ in } \mathcal{B}.$$
2. $\mathbb{K}$ *has a Kolmogorov decomposition in the sense of* [18]*, i.e., there exists an* $(\mathcal{A}, \mathcal{B})$*-correspondence* E *and a mapping* $\omega \mapsto k_\omega$ *from* Ω *into* E *such that*
$$\mathbb{K}(\omega', \omega)[a] = \langle a \cdot k_\omega,\, k_{\omega'} \rangle_E \text{ for all } a \in \mathcal{A}.$$
3. $\mathbb{K}$ *is the reproducing kernel for an* $(\mathcal{A}, \mathcal{B})$*-reproducing kernel correspondence* $E = E(\mathbb{K})$*, i.e., there is an* $(\mathcal{A}, \mathcal{B})$*-correspondence* $E = E(\mathbb{K})$ *whose elements are* $\mathcal{B}$*-valued functions on* $\Omega \times \mathcal{A}$ *such that the function* $k_\omega \colon (\omega', a') \mapsto \mathbb{K}(\omega', \omega)[a']$ *is in* $E(\mathcal{K})$ *for each* $\omega \in \Omega$ *and has the reproducing property*
$$\langle a \cdot f, k_\omega \rangle_{E(\mathbb{K})} = \langle f, a^* \cdot k_\omega \rangle_{E(\mathbb{K})} = f(\omega, a) \text{ for all } \omega \in \Omega \quad \text{and} \quad a \in \mathcal{A}$$
where $a^* \cdot k_\omega$ *is given by*
$$(a^* \cdot k_\omega)(\omega', a') = \mathbb{K}(\omega', \omega)[a' a^*]. \tag{3.10}$$

Proof. For the equivalence of (1) and (2), we refer to Theorem 3.2.3 in [18]. The argument in the paragraph preceding the statement of the theorem shows that (3) $\Longrightarrow$ (1). To see that (2) $\Longrightarrow$ (3), assume that E is an $(\mathcal{A}, \mathcal{B})$-correspondence as in (2). Without loss of generality we may assume that
$$E = \overline{\text{span}}\{a \cdot k_\omega b \colon a \in \mathcal{A}, \omega \in \Omega, b \in \mathcal{B}\}. \tag{3.11}$$
We view elements f of E as $\mathcal{B}$-valued functions on $\Omega \times \mathcal{A}$ by defining
$$f(\omega, a) = \langle a \cdot f, k_\omega \rangle_E \text{ for each } \omega \in \Omega \text{ and } a \in \mathcal{A}.$$
The nondegeneracy assumption (3.11) says that
$$f(\omega, a) b = 0 \text{ for all } a \in \mathcal{A},\, b \in \mathcal{B},\, \omega \in \Omega \Longrightarrow f = 0 \text{ in } E.$$
Hence the map $f \mapsto f(\cdot, \cdot)$ is injective. Finally (3.10) holds by definition. □

We now tailor this general theorem to the case where $\mathcal{B} = \mathcal{L}(\mathcal{E})$ for a Hilbert space $\mathcal{E}$. Note that $\mathcal{E}$ is a $(\mathcal{L}(\mathcal{E}), \mathbb{C})$-correspondence, i.e., a Hilbert space with a $*$-representation $b \mapsto \varphi(b) \in \mathcal{L}(\mathcal{E})$ of $\mathcal{L}(\mathcal{E})$ (namely, the identity representation). Hence, given that E is an $(\mathcal{A}, \mathcal{L}(\mathcal{E}))$-correspondence, we may form the tensor product $E \otimes_{\mathcal{L}(\mathcal{E})} \mathcal{E}$ to obtain an $(\mathcal{A}, \mathbb{C})$-correspondence, i.e., a Hilbert space which we will denote by $\mathcal{H}$ equipped with an $\mathcal{L}(\mathcal{H})$-valued $*$-representation $\pi \colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ of $\mathcal{A}$. Similarly, if we view $\mathcal{B} = \mathcal{L}(\mathcal{E})$ as a $(\mathcal{L}(\mathcal{E}), \mathcal{L}(\mathcal{E}))$-correspondence, we may form the tensor product $\mathcal{L}(\mathcal{E})\otimes\mathcal{E}$ to arrive at the Hilbert space $\mathcal{E}$, via the balancing (3.4), viewed as a $(\mathcal{L}(\mathcal{E}), \mathbb{C})$-correspondence. Let us suppose also that E is a reproducing kernel correspondence. Then via the formula $f\otimes e \in E\otimes\mathcal{E} \mapsto f(\omega, a)\otimes e \in \mathcal{L}(\mathcal{E})\otimes\mathcal{E}$ for each $\omega \in \Omega$ and $a \in \mathcal{A}$ extended via linearity and continuity to the whole space $E\otimes\mathcal{E}$, we may view each $\mathbf{f} \in \mathcal{H} = E\otimes\mathcal{E}$ as a $\mathcal{E}$-valued function on $\Omega\times\mathcal{A}$ such that point-evaluation $\mathbf{f} \mapsto \mathbf{f}(\omega, a)$ is continuous, i.e., $\mathcal{H}$ is a reproducing kernel Hilbert space of vector-valued functions on $\Omega \times \mathcal{A}$, but with the additional wrinkle that there is also a representation $a \mapsto \pi(a)$ of $\mathcal{A}$ on $\mathcal{H}$ with $\pi(a)(f \otimes e) = (a \cdot f) \otimes e$ such that

$$(\pi(a)(f \otimes e))(\omega', a') = f(\omega', a'a) \otimes e$$

with reproducing kernel (in the sense of a vector-valued reproducing kernel Hilbert space) $K(\cdot, \cdot)$ of the special form

$$K((\omega', a'), (\omega, a)) = \mathbb{K}(\omega', \omega)[a^*a']$$

for a completely positive kernel $\mathbb{K}\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{E}))$: for $\mathbf{f} \in \mathcal{H}(\mathbb{K})$, $e \in \mathcal{E}$ and $(\omega, a) \in \Omega \times \mathcal{A}$,

$$\langle \mathbf{f}, \mathbb{K}(\cdot, \omega)[a]e\rangle_{\mathcal{H}} = \langle \mathbf{f}(\omega, a), e\rangle_{\mathcal{E}}$$

where $\mathbb{K}$ is completely positive. This leads us to an alternative reproducing-kernel interpretation of a completely positive kernel $\mathbb{K}\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})$ for the case where $\mathcal{B} = \mathcal{L}(\mathcal{E})$ for a Hilbert space $\mathcal{E}$.

Theorem 3.6. *Suppose that $\mathcal{A}$ is a C^*-algebra, $\mathcal{E}$ is a Hilbert space and that a function $\mathbb{K}\colon \Omega\times\Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{E}))$ is given. The following conditions are equivalent.*

1. *The function $\mathbb{K}$ is a completely positive kernel in the sense that*

$$\sum_{i,j=1}^{N} \langle \mathbb{K}(\omega_i, \omega_j)[a_i^* a_j]e_j, e_i\rangle \geq 0$$

for all finite collections $\omega_1, \dots, \omega_N \in \Omega$, $a_1, \dots, a_N \in \mathcal{A}$ and $e_1, \dots, e_N \in \mathcal{E}$ for $N = 1, 2, \dots$.

2. *The kernel $\mathbb{K}$ has a Kolmogorov decomposition: there is a Hilbert space $\mathcal{H}$ together with a $*$-representation $\pi\colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$ of $\mathcal{A}$ and a mapping $H\colon \Omega \to \mathcal{L}(\mathcal{H}, \mathcal{E})$ so that*

$$\mathbb{K}(\omega', \omega)[a] = H(\omega')\pi(a)H(\omega)^*.$$

3. *There is a $(\mathcal{A}, \mathbb{C})$-correspondence, i.e., a Hilbert space $\mathcal{H} = \mathcal{H}(\mathbb{K})$ together with a $*$-representation $a \mapsto \pi(a) \in \mathcal{L}(\mathcal{H})$ of $\mathcal{A}$ whose elements are $\mathcal{E}$-valued functions on $\Omega \times \mathcal{A}$ such that:*
   (a) *The $*$-representation π is given by*

$$(\pi(a)\mathbf{f})(\omega', a') = \mathbf{f}(\omega', a'a).$$

   (b) *The function $k_\omega\colon \Omega \times \mathcal{A} \to \mathcal{L}(\mathcal{E})$ given by*

$$k_\omega(\omega', a')\colon e \mapsto \mathbb{K}(\omega', \omega)[a']e$$

is such that $k_\omega e \in \mathcal{H}(\mathbb{K})$ for each $\omega \in \Omega$ and $e \in \mathcal{E}$ and has the reproducing kernel property:

$$\langle \mathbf{f}, \pi(a)^* k_\omega e \rangle_{\mathcal{H}(\mathbb{K})} = \langle \mathbf{f}(\omega, a), e \rangle_{\mathcal{E}}.$$

Let us say that the object described in part (3) of Theorem 3.6 a *reproducing kernel Hilbert correspondence* (over the C^*-algebra $\mathcal{A}$ with values in the coefficient space $\mathcal{E}$).

Remark 3.7. If $\mathcal{H}(\mathbb{K})$ is a reproducing kernel Hilbert correspondence space as in part (3) of Theorem 3.6, a special situation occurs if the coefficient space $\mathcal{E}$ is also equipped with a $*$-representation $\pi_{\mathcal{E}} \colon \mathcal{A} \to \mathcal{L}(\mathcal{E})$. In this case it may or may not happen that point evaluation is an $\mathcal{A}$-module map, i.e., that

$$(a \cdot f)(\omega', a') = a \cdot f(\omega', a') \text{ or equivalently } (\pi(a) f)(\omega', a') = \pi_{\mathcal{E}}(a) f(\omega', a'). \tag{3.12}$$

When (3.12) does occur and if also $\mathcal{A}$ has a unit $1_{\mathcal{A}}$, one can show that the associated completely positive kernel $\mathbb{K}(\omega, \omega')[a]$ has the special property

$$\mathbb{K}(\omega, \omega')[a^* a'] = \pi_{\mathcal{E}}(a)^* \mathbb{K}(\omega, \omega')[1_{\mathcal{A}}] \pi_{\mathcal{E}}(a') \tag{3.13}$$

and hence complete positivity of $\mathbb{K}$ reduces to standard (Aronszajn) positivity for the kernel $K_0 \colon \Omega \times \Omega \to \mathcal{L}(\mathcal{E})$ given by

$$K_0(\omega, \omega') = \mathbb{K}(\omega, \omega')[1_{\mathcal{A}}].$$

Indeed, the computation

$$\begin{aligned}
\langle \mathbb{K}(\omega, \omega')[a^* a'] e', e \rangle_{\mathcal{E}} &= \langle a^* a' \cdot k_{\omega'} e',\, k_\omega e \rangle_{\mathcal{H}(\mathbb{K})} \\
&= \langle (a^* a' \cdot k_{\omega'} e')(\omega),\, e \rangle_{\mathcal{E}} \\
&= \langle a^* a' \cdot (k_{\omega'} e')(\omega),\, e \rangle_{\mathcal{E}} \text{ (by assumption (3.12))} \\
&= \langle a' (k_{\omega'} e')(\omega), ae \rangle_{\mathcal{E}} \\
&= \langle (a' \cdot k_{\omega'} e')(\omega), ae \rangle_{\mathcal{E}} \text{ (by (3.12) again)} \\
&= \langle a' \cdot k_{\omega'} e', k_\omega(ae) \rangle_{\mathcal{H}(\mathbb{K})} \\
&= \langle \mathbb{K}(\omega, \omega')[a'] e', ae \rangle_{\mathcal{E}} \\
&= \langle a^* \mathbb{K}(\omega, \omega')[a'] e', e \rangle_{\mathcal{E}}
\end{aligned}$$

shows that

$$\mathbb{K}(\omega, \omega')[a^* a'] = a^* \mathbb{K}(\omega, \omega')[a']. \tag{3.14}$$

On the other hand, the positive-kernel property of the kernel

$$((\omega, a), (\omega', a')) \mapsto K((\omega, a), (\omega', a')) := \mathbb{K}(\omega, \omega')[a^* a']$$

implies that K is Hermitian, i.e., $K((\omega, a), (\omega', a')) = K((\omega', a'), (\omega, a))^*$, i.e.,

$$\mathbb{K}(\omega, \omega')[a^* a'] = \left(\mathbb{K}(\omega', \omega)[a'^* a]\right)^*.$$

In particular,

$$\begin{aligned}\mathbb{K}(\omega,\omega')[a'] &= (\mathbb{K}(\omega',\omega)[a'^*])^* \\ &= (a'^*\mathbb{K}(\omega',\omega)[1_{\mathcal{A}}])^* \quad \text{(by (3.14))} \\ &= \mathbb{K}(\omega,\omega')[1_{\mathcal{A}}]a'\end{aligned}$$

and hence also

$$\mathbb{K}(\omega,\omega')[a'] = \mathbb{K}(\omega,\omega')[1_{\mathcal{A}}]a'. \tag{3.15}$$

Combining (3.14) and (3.15) gives (3.13) as claimed.

4. Function-theoretic operator theory associated with a correspondence E

In this section we obtain the analogues of Hardy spaces, Toeplitz operators, Z-transform and Schur class attached to a $\mathcal{A}$-W^*-correspondence E together with a $*$-representation σ of $\mathcal{A}$. These results flesh out more fully the function-theoretic aspects of the work of Muhly-Solel [31, 33, 36].

4.1. Hardy Hilbert spaces associated with a correspondence E

In this section we shall consider the situation where $\mathcal{A} = \mathcal{B}$; we abbreviate the term $(\mathcal{A},\mathcal{A})$-correspondence to simply $\mathcal{A}$-correspondence. We also now restrict our attention to the case where $\mathcal{A}$ is a von Neumann algebra and let E be a $\mathcal{A}$-W^*-correspondence. This means that E is a $\mathcal{A}$-correspondence which is also *self-dual* in the sense that any right $\mathcal{A}$-module map $\rho\colon E \to \mathcal{A}$ is given by taking the inner product against some element e_ρ of E: $\rho(e) = \langle e, e_\rho\rangle_E \in \mathcal{A}$. Moreover, the space $\mathcal{L}^a(E)$ of adjointable operators on the W^*-correspondence E is in fact a W^*-algebra, i.e., is the abstract version of a von Neumann algebra with an ultra-weak topology (see [33]).

Since E is a $\mathcal{A}$-correspondence, we may use Definition 3.1 to define the self-tensor product $E^{\otimes 2} = E \otimes_{\mathcal{A}} E$ which is again an $\mathcal{A}$-correspondence, and, inductively, an $\mathcal{A}$-correspondence $E^{\otimes n} = E \otimes_{\mathcal{A}} (E^{\otimes(n-1)})$ for each $n = 1, 2, \dots$. If we use $a \mapsto \varphi(a)$ to denote the left $\mathcal{A}$-action $\varphi(a)e = a\cdot e$ on E, we denote the left $\mathcal{A}$-action on $E^{\otimes n}$ by $\varphi^{(n)}$:

$$\varphi^{(n)}(a)\colon \xi_n \otimes \xi_{n-1} \otimes \cdots \otimes \xi_1 \mapsto (\varphi(a)\xi_n) \otimes \xi_{n-1} \otimes \cdots \otimes \xi_1.$$

Note that, using the notation in (3.5), we may write $\varphi^{(n)}(a) = \varphi(a) \otimes I_{E^{\otimes n-1}}$. We formally set $E^{\otimes 0} = \mathcal{A}$. Then the Fock space $\mathcal{F}^2(E)$ is defined to be

$$\mathcal{F}^2(E) = \oplus_{n=0}^\infty E^{\otimes n} \tag{4.1}$$

and is also an $\mathcal{A}$-correspondence. We denote the left $\mathcal{A}$-action on $\mathcal{F}(E)$ by φ_∞:

$$\varphi_\infty(a)\colon\ \oplus_{n=0}^\infty \xi^{(n)} \mapsto \oplus_{n=0}^\infty(\varphi^{(n)}(a)\xi^{(n)}) \text{ for } \oplus_{n=0}^\infty \xi^{(n)} \in \bigoplus_{n=0}^\infty E^{\otimes n}, \tag{4.2}$$

or, more succinctly,

$$\varphi_\infty(a) = \mathrm{diag}(a, \varphi^{(1)}(a), \varphi^{(2)}(a), \dots).$$

In addition to the von Neumann algebra $\mathcal{A}$ and the $\mathcal{A}$-correspondence E, suppose that we are also given an auxiliary Hilbert space $\mathcal{E}$ and a nondegenerate $*$-homomorphism $\sigma\colon \mathcal{A} \to \mathcal{L}(\mathcal{E})$; as this will be the setting for much of the analysis to follow, we refer to such a pair (E, σ) as a *correspondence-representation pair.* Then the Hilbert space $\mathcal{E}$ equipped with σ becomes an $(\mathcal{A}, \mathbb{C})$-correspondence with left $\mathcal{A}$-action given by σ:

$$a \cdot y = \sigma(a)y \text{ for all } a \in \mathcal{A} \quad \text{and} \quad y \in \mathcal{E}.$$

We let $E \otimes_\sigma \mathcal{E}$ be the associated tensor product $(\mathcal{A}, \mathbb{C})$-correspondence $E \otimes_{\mathcal{A}} \mathcal{E}$ as in Definition 3.1. As $\mathcal{F}^2(E)$ is also an $\mathcal{A}$-correspondence, we may also form the $(\mathcal{A}, \mathbb{C})$-correspondence

$$\mathcal{F}^2(E, \sigma) := \mathcal{F}^2(E) \otimes_\sigma \mathcal{E} = \bigoplus_{n=0}^{\infty} (E^{\otimes n} \otimes_\sigma \mathcal{E}),$$

with left $\mathcal{A}$-action given by the $*$-representation

$$\varphi_{\infty,\sigma}(a) = \varphi_\infty(a) \otimes I_{\mathcal{E}}.$$

It turns out that $\mathcal{F}^2(E, \sigma)$ is also a $(\sigma(\mathcal{A})', \mathbb{C})$-correspondence, where $\sigma(\mathcal{A})' \subset \mathcal{L}(\mathcal{E})$ denotes the commutant of $\sigma(\mathcal{A})$:

$$\sigma(\mathcal{A})' = \{b \in \mathcal{L}(\mathcal{E})\colon b\sigma(a) = \sigma(a)b \text{ for all } a \in \mathcal{A}\}, \tag{4.3}$$

and where the left $\sigma(\mathcal{A})'$-action is given by the $*$-representation $\iota_{\infty,\sigma}$ of $\sigma(\mathcal{A})'$ on $\mathcal{L}(\mathcal{F}^2(E, \sigma))$:

$$\iota_{\infty,\sigma}(b) = I_{\mathcal{F}^2(E)} \otimes b \text{ for each } b \in \sigma(\mathcal{A})', \tag{4.4}$$

using the notation in (3.5). Note that $b \in \mathcal{L}(\mathcal{E})$ is in $\sigma(\mathcal{A})'$ precisely when b is an $\mathcal{A}$-module map, so that $I_{\mathcal{F}^2(E)} \otimes b$ is a well-defined operator on $\mathcal{F}^2(E, \sigma)$. Moreover, $\varphi_{\infty,\sigma}(a)$ commutes with $\iota_{\infty,\sigma}(b)$ for each $a \in \mathcal{A}$ and $b \in \sigma(\mathcal{A})'$ since

$$\varphi_{\infty,\sigma}(a)\iota_{\infty,\sigma}(b) = \varphi_\infty(a) \otimes b = \iota_{\infty,\sigma}(b)\varphi_{\infty,\sigma}(a).$$

Thus $\iota_{\infty,\sigma}(b)$ is a $\mathcal{A}$-module map for each $b \in \sigma(\mathcal{A})'$ and $\varphi_{\infty,\sigma}(a)$ is a $\sigma(\mathcal{A})'$-module map for each $a \in \mathcal{A}$.

We denote by E^σ the set of all bounded linear operators $\mu\colon \mathcal{E} \to E \otimes_\sigma \mathcal{E}$ which are also $\mathcal{A}$-module maps:

$$E^\sigma = \{\mu\colon \mathcal{E} \to E \otimes_\sigma \mathcal{E}\colon \mu\sigma(a) = (\varphi(a) \otimes I_{\mathcal{E}})\mu\}, \tag{4.5}$$

and $(E^\sigma)^*$ for the set of adjoints (which are also $\mathcal{A}$-module maps):

$$(E^\sigma)^* = \{\eta : E \otimes_\sigma \mathcal{E} \to \mathcal{E}\colon \eta^* \in E^\sigma\}. \tag{4.6}$$

More generally, for a given $\eta \in (E^\sigma)^*$, we may define operators $\eta^n\colon E^{\otimes n} \otimes_\sigma \mathcal{E} \to \mathcal{E}$ (*generalized powers*) by

$$\eta^n = \eta(I_E \otimes \eta) \cdots (I_{E^{\otimes n-1}} \otimes \eta)$$

where we use the identification

$$E^{\otimes n} \otimes_\sigma \mathcal{E} = E^{\otimes n-1} \otimes_{\mathcal{A}} (E \otimes_\sigma \mathcal{E})$$

in these definitions. We also set $\eta^0 = I_{\mathcal{E}} \in \mathcal{L}(\mathcal{E})$. Again the fact that η is an $\mathcal{A}$-module map ensures that $I_{E^{\otimes k}} \otimes \eta$ is a well-defined operator in $\mathcal{L}(E^{\otimes k+1} \otimes_\sigma \mathcal{E}, E^{\otimes k} \otimes_\sigma \mathcal{E})$. The defining $\mathcal{A}$-module property of η in (4.6) then extends to the generalized powers η^n in the form

$$\eta^n(\varphi^{(n)}(a) \otimes I_{\mathcal{E}}) = \sigma(a)\eta^n, \tag{4.7}$$

i.e., η^n is also an $\mathcal{A}$-module map.

Denote by $\mathbb{D}((E^\sigma)^*)$ the set of strictly contractive elements of $(E^\sigma)^*$:

$$\mathbb{D}((E^\sigma)^*) = \{\eta \in (E^\sigma)^* \colon \|\eta\| < 1\}.$$

Then, for $\eta \in \mathbb{D}((E^\sigma)^*)$ and $b \in \sigma(\mathcal{A})'$, we may define a bounded operator $f \mapsto f^\wedge(\eta, b)$ from $\mathcal{F}^2(E,\sigma)$ into $\mathcal{E}$ by

$$f^\wedge(\eta, b) = \sum_{n=0}^{\infty} \eta^n(\iota_{\infty,\sigma}(b)f)_n = \sum_{n=0}^{\infty} \eta^n(I_{E^{\otimes n}} \otimes b)f_n \text{ if } f = \oplus_{n=0}^{\infty} f_n. \tag{4.8}$$

Note that the fact that $\|\eta\| < 1$ guarantees that the series in (4.8) converges. The $\mathcal{A}$-module properties of $\iota_{\infty,\sigma}(b)$ and each generalized power η^n (see (4.7)) for given $b \in \sigma(\mathcal{A})'$ and $\eta \in \mathbb{D}((E^\sigma)^*)$ imply that the point-evaluation $f \mapsto f^\wedge(\eta, b)$ is also an $\mathcal{A}$-module map:

$$(\varphi_{\infty,\sigma}(a)f)^\wedge(\eta, b) = \sigma(a) f^\wedge(\eta, b).$$

However, the point-evaluation $f \mapsto f^\wedge(\eta, b)$ is not a $\sigma(\mathcal{A})'$-module map, i.e., there is no guarantee for the general validity of the identity $(\iota_{\infty,\sigma}(b)f)^\wedge(\eta', b') = bf^\wedge(\eta', b')$, but rather we have the property

$$(\iota_{\infty,\sigma}(b))f)^\wedge(\eta', b') = f^\wedge(\eta', b'b).$$

We denote the space of all $\mathcal{E}$-valued functions on $\mathbb{D}((E^\sigma)^*) \times \sigma(\mathcal{A})'$ of the form $(\eta, b) \mapsto f^\wedge(\eta, b)$ for some $f \in \mathcal{F}^2(E,\sigma)$ by $H^2(E,\sigma)$ with norm $\|f^\wedge\|_{H^2(E,\sigma)}$ chosen so as to make the map $f \mapsto f^\wedge$ a coisometry from $\mathcal{F}^2(E,\sigma)$ to $H^2(E,\sigma)$:

$$H^2(E,\sigma) = \{f^\wedge \colon f \in \mathcal{F}^2(E,\sigma)\} \text{ with } \|f^\wedge\|_{H^2(E,\sigma)} = \|P_{(\operatorname{Ker}\Phi)^\perp} f\|_{\mathcal{F}^2(E,\sigma)}$$

where we denote by Φ (the generalized *Fourier* or Z-transform for this setting) the transformation from $\mathcal{F}^2(e,\sigma)$ into $H^2(E,\sigma)$ given by

$$\Phi \colon f \mapsto f^\wedge. \tag{4.9}$$

Then we have the following result.

Theorem 4.1. *The space $H^2(E,\sigma)$ is a reproducing kernel Hilbert correspondence $\widetilde{\mathcal{H}}(\mathbb{K})$ (as in part (3) of Theorem 3.6) over $\sigma(\mathcal{A})'$ consisting of $\mathcal{E}$-valued functions on $\mathbb{D}((E^\sigma)^*) \times \sigma(\mathcal{A})'$ with the $*$-representation of $\sigma(\mathcal{A})'$ given by*

$$(b \cdot f^\wedge)(\eta', b') = (\iota_{\infty,\sigma}(b)f)^\wedge(\eta', b') \text{ for } b \in \sigma(A)'. \tag{4.10}$$

The completely positive kernel $\mathbb{K}$ *associated with* $H^2(E,\sigma)$ *as in Theorem* 3.6

$$\mathbb{K}_{E,\sigma}\colon \mathbb{D}((E^\sigma)^*) \times \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(\sigma(\mathcal{A})', \mathcal{L}(\mathcal{E}))$$

is the Szegö kernel *for our setting given by*

$$\mathbb{K}_{E,\sigma}(\eta,\zeta)[b] = \sum_{n=0}^{\infty} \eta^n (I_{E^{\otimes n}} \otimes b)(\zeta^n)^* \ \textit{for } b \in \sigma(\mathcal{A})'. \tag{4.11}$$

Proof. Define $\Phi\colon \mathcal{F}^2(E,\sigma) \to H^2(E,\sigma)$ as in (4.9). By the definition of the norm on $H^2(E,\sigma)$, Φ is a coisometry. For each $b \in \sigma(\mathcal{A})'$ and $\eta \in \mathbb{D}((E^\sigma)^*)$, define an associated *controllability operator*[2] $\mathcal{C}_{b,\eta}\colon \mathcal{F}^2(E,\sigma) \to \mathcal{E}$ by

$$\mathcal{C}_{b,\eta}\colon f \mapsto f^\wedge(\eta, b) \text{ if } f \in \mathcal{F}^2(E,\sigma).$$

By definition,

$$\operatorname{Ker}\Phi = \bigcap_{b\in\sigma(\mathcal{A})',\eta\in\mathbb{D}((E^\sigma)^*)} \operatorname{Ker}\mathcal{C}_{b,\eta}.$$

The initial space of the coisometry Φ is the orthogonal complement of its kernel, namely

$$(\operatorname{Ker}\Phi)^\perp = \overline{\operatorname{span}}\{\operatorname{Ran}\mathcal{C}_{b,\eta}^*\colon b \in \sigma(\mathcal{A})',\ \eta \in \mathbb{D}((E^\sigma)^*)\},$$

where the *observability operator* $\mathcal{C}_{b,\eta}^*$ is given by

$$\mathcal{C}_{b,\eta}^*\colon e \mapsto \oplus_{n=0}^{\infty} (I_{E^{\otimes n}} \otimes b^*)(\eta^n)^* e \ \in \mathcal{F}^2(E) \otimes_\sigma \mathcal{E}.$$

We compute

$$\begin{aligned}\langle f^\wedge(\zeta,b), e\rangle_{\mathcal{E}} &= \langle \mathcal{C}_{b,\zeta} f, e\rangle_{\mathcal{E}}\\ &= \langle f, \mathcal{C}_{b,\zeta}^* e\rangle_{\mathcal{F}^2(E,\sigma)}\\ &= \langle f^\wedge, \Phi(\mathcal{C}_{b,\zeta}^* e)\rangle_{H^2(E,\sigma)}\\ &= \langle f^\wedge, b^* \cdot \Phi(\mathcal{C}_{I_{\mathcal{E}},\zeta}^* e)\rangle_{H^2(E,\sigma)},\end{aligned}$$

where we use the fact seen above that $\mathcal{C}_{b,\zeta}^* e$ is in the initial space of Φ and that $\Phi((I_{\mathcal{F}^2}(E) \otimes b) f) = b(\Phi f)$ for each $b \in \sigma(\mathcal{A})'$ and $f \in \mathcal{F}^2(e,\sigma)$. Hence the operator

$$k_{E,\sigma;\zeta} := \Phi \mathcal{C}_{I_{\mathcal{E}},\zeta}^* : \mathcal{E} \to H^2(E,\sigma)$$

has the reproducing property for $H^2(E,\sigma)$; see part (3.b) in Theorem 3.6. Since Φ is a coisometry and $I_{\mathcal{E}} \in \sigma(\mathcal{A})'$, we obtain that the reproducing kernel $\mathbb{K}_{E,\sigma}$ is necessarily given by

$$\begin{aligned}\mathbb{K}_{E,\sigma}(\eta,\zeta)[b] &= b \cdot k_{E,\sigma;\zeta}(\eta)\\ &= \mathcal{C}_{b,\eta}\Phi^*\Phi\mathcal{C}_{I_{\mathcal{E}},\zeta}^*\\ &= \sum_{n=0}^{\infty} \eta^n (I_{E^{\otimes n}} \otimes b)(\zeta^n)^*\end{aligned}$$

in agreement with (4.11). □

[2]The terminology is motivated by connections with system theory; for a systematic account for the Drury-Arveson and free-semigroup algebra settings, we refer to [10].

From the proof of Theorem 4.1 we see that we have the identification

$$b^* \cdot k_{E,\sigma;\eta} e = \mathcal{C}^*_{b,\eta} e = \oplus_{n=0}^{\infty} (I_{E^{\otimes n}} \otimes b^*) \eta^{n*} e$$

and the *initial space* for the coisometry $\Phi \colon \mathcal{F}(E) \otimes_\sigma \mathcal{E} \to H^2(E,\sigma)$ can be identified as

$$[\mathcal{F}(E) \otimes_\sigma \mathcal{E}]_{\text{initial}} = \overline{\text{span}}\{b \cdot k_{E,\sigma;\eta} e \colon b \in \sigma(\mathcal{A})',\ \eta \in \mathbb{D}((E^\sigma)^*), e \in \mathcal{E}\}. \tag{4.12}$$

4.2. Analytic Toeplitz algebras associated with a correspondence E

Given an $\mathcal{A} - W^*$-correspondence E, we let $\mathcal{F}^2(E)$ be the associated Fock space as in (4.1). We have already defined the $*$-representation of $\mathcal{A}$ to $\mathcal{L}^a(\mathcal{F}^2(E))$ given by $a \mapsto \varphi_\infty(a)$ as in (4.2). If we view operators on $\mathcal{F}^2(E)$ as matrices induced by the decomposition $\mathcal{F}^2(E) = \bigoplus_{n=0}^{\infty} E^{\otimes n}$ of $\mathcal{F}^2(E)$, we see that each $\varphi_\infty(a)$ has a diagonal representation $\varphi_\infty(a) = \text{diag}_{n=0,1,\dots} \varphi^{(n)}(a)$. In addition to the operators $\varphi_\infty(a) \in \mathcal{L}^a(\mathcal{F}^2(E))$, we introduce the so-called *creation operators* on $\mathcal{F}^2(E)$ given, for each $\xi \in E$, by the subdiagonal (or shift) block matrix

$$T_\xi = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ T_\xi^{(0)} & 0 & 0 & \cdots \\ 0 & T_\xi^{(1)} & 0 & \cdots \\ \ddots & \ddots & \ddots & \end{bmatrix}$$

where the block entry $T_\xi^{(n)} \colon E^{\otimes n} \to E^{\otimes n+1}$ is given by

$$T_\xi^{(n)} \colon \xi_n \otimes \cdots \otimes \xi_1 \mapsto \xi \otimes \xi_n \otimes \cdots \otimes \xi_1.$$

The operator T_ξ is also in $\mathcal{L}^a(\mathcal{F}^2(E))$. In summary, both T_ξ and $\varphi_\infty(a)$ are $\mathcal{A}$-module maps with respect to the right $\mathcal{A}$-action on $\mathcal{F}^2(E)$ for each $\xi \in E$ and $a \in \mathcal{A}$. Moreover, one easily checks that

$$\varphi_\infty(a) T_\xi = T_{a\xi} = T_{\varphi(a)\xi} \quad \text{and} \quad T_\xi \varphi_\infty(a) = T_{\xi a} \quad \text{for each } a \in \mathcal{A} \text{ and } \xi \in E.$$

We let $\mathcal{F}^\infty(E)$ denote the weak-$*$ closed algebra generated by the collection of operators

$$\{\varphi_\infty(a), T_\xi \colon a \in \mathcal{A} \quad \text{and} \quad \xi \in E\}$$

in the W^*-algebra $\mathcal{L}^a(\mathcal{F}(E))$ – we prefer this notation over the notation $H^\infty(E)$ used for this object in [31, 33].

Suppose now that we are also given a $*$-representation σ of $\mathcal{A}$ on a Hilbert space $\mathcal{E}$. Rather than the algebra $\mathcal{F}^\infty(E)$ of adjointable operators on the $\mathcal{A}$-correspondence $\mathcal{F}^2(E)$, our main focus of interest will be on the algebra $\mathcal{F}^\infty(E) \otimes I_\mathcal{E}$ of all operators on the Hilbert space $\mathcal{F}^2(E,\sigma)$ of the form $R = T \otimes I_\mathcal{E}$ with $T \in \mathcal{F}^\infty(E)$ acting on the Hilbert space $\mathcal{F}^2(E,\sigma)$. Note that the operator $R = T \otimes I_\mathcal{E}$ is properly defined since T is an $\mathcal{A}$-module map with respect to the right $\mathcal{A}$-action on $\mathcal{F}^2(E)$. For convenience we shall use the abbreviated notation

$$\mathcal{F}^\infty(E,\sigma) = \mathcal{F}^\infty(E) \otimes I_\mathcal{E},$$

and

$$\varphi_{\infty,\sigma}(a) = \varphi_\infty(a) \otimes I_\mathcal{E} \quad \text{and} \quad T_{\xi,\sigma} = T_\xi \otimes I_\mathcal{E} \quad \text{for all } a \in \mathcal{A} \text{ and } \xi \in E.$$

The algebra $\mathcal{F}^\infty(E,\sigma)$ can also be described as the weak-$*$ closed algebra generated by the collection of operators

$$\{\varphi_{\infty,\sigma}(a),\, T_{\xi,\sigma} \colon a \in \mathcal{A},\, \xi \in E\}. \tag{4.13}$$

The following alternative characterization of $\mathcal{F}^\infty(E,\sigma)$ will be useful. Here we define E^σ and $\sigma(\mathcal{A})'$ as in (4.5) and (4.3). Note that each element μ of E^σ induces a *dual creation operator* $T^d_{\mu,\sigma}$ in $\mathcal{L}^a(\mathcal{F}^2(E,\sigma))$ given by

$$T^d_{\mu,\sigma} = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ T^{d,(0)}_{\mu,\sigma} & 0 & 0 & \cdots \\ 0 & T^{d,(1)}_{\mu,\sigma} & 0 & \cdots \\ \ddots & \ddots & \ddots & \end{bmatrix}$$

where $T^{d,(n)}_{\mu,\sigma} \colon E^{\otimes n} \otimes_\sigma \mathcal{E} \to E^{\otimes n+1} \otimes_\sigma \mathcal{E}$ is given by

$$T^{d,(n)}_{\mu,\sigma} \colon \xi_n \otimes \cdots \otimes \xi_1 \otimes e \mapsto \xi_n \otimes \cdots \otimes \xi_1 \otimes \mu e$$

where as usual we make the identification

$$E^{\otimes n} \otimes (E \otimes_\sigma \mathcal{E}) = E^{\otimes n+1} \otimes_\sigma \mathcal{E}.$$

Using the notation in (3.5) we can write $T^d_{\mu,\sigma} = I_{\mathcal{F}^2(E)} \otimes \mu$, where we identify $\mathcal{F}^2(E) \otimes_\mathcal{A} E$ with $\mathcal{F}^2(E)$, which makes sense because μ is an $\mathcal{A}$-module map. Also recall that $\iota_{\infty,\sigma}$ in (4.4) defines a $*$-representation of $\sigma(\mathcal{A})'$ on $\mathcal{F}^2(E,\sigma)$.

Proposition 4.2. *An operator $R \in \mathcal{L}(\mathcal{F}^2(E,\sigma))$ is in $\mathcal{F}^\infty(E,\sigma)$ if and only if R commutes with each of the operators $I_{\mathcal{F}(E)} \otimes_\sigma b$ and $T^d_{\mu,\sigma}$ for $b \in \sigma(\mathcal{A})'$ and $\mu \in E^\sigma$. Consequently, the operator $R \in \mathcal{L}(\mathcal{F}^2(E,\sigma))$ with infinite block-matrix representation*

$$R = [R_{i,j}]_{i,j=0,1,2,\dots} \quad \textit{where} \quad R_{i,j} \colon E^{\otimes j} \otimes_\sigma \mathcal{E} \to E^{\otimes i} \otimes_\sigma \mathcal{E}$$

is in $\mathcal{F}^\infty(E,\sigma)$ if and only if R is lower triangular ($R_{i,j} = 0$ for $i < j$) and for $i \geq j$ $R_{i,j}$ satisfies the following compatibility (Toeplitz-like) conditions:

$$R_{i,j}(I_{E^{\otimes j}} \otimes b) = (I_{E^{\otimes i}} \otimes b) R_{i,j} \text{ for all } b \in \sigma(\mathcal{A})', \tag{4.14}$$

$$R_{i+1,j+1}(I_{E^{\otimes j}} \otimes \mu) = (I_{E^{\otimes i}} \otimes \mu) R_{i,j} \text{ for all } \mu \in E^\sigma. \tag{4.15}$$

and hence, inductively,

$$R_{i,j}\mu^j = (I_{E^{\otimes i-j}} \otimes \mu^j) R_{i-j,0}, \tag{4.16}$$

where $\mu^j = ((\mu^*)^j)^*$, with $(\mu^*)^j$ the generalized power of $\mu^* \in (E^\sigma)^*$.

Proof. The first part follows from Theorem 3.9 of [33]. The second part is then a straightforward translation of these commutativity conditions to expressions involving the block entries. □

Taking the cue from Proposition 4.2, we view elements R of $\mathcal{F}^\infty(E,\sigma)$ as the *analytic Toeplitz operators* for this Fock-space/correspondence setting.

While it is in general not the case that $R(T^d_{\mu,\sigma})^* = (T^d_{\mu,\sigma})^* R$ for $R \in \mathcal{F}^\infty(E,\sigma)$ and $\mu \in E^\sigma$, this is almost the case as is made precise in the following proposition.

Proposition 4.3. *For $R \in \mathcal{F}^\infty(E,\sigma)$ and $\mu \in E^\sigma$, we have*

$$R(T^d_{\mu,\sigma})^*|_{\oplus_{n=1}^\infty E^{\otimes n}\otimes\mathcal{E}} = (T^d_{\mu,\sigma})^* R|_{\oplus_{n=1}^\infty E^{\otimes n}\otimes\mathcal{E}}, \tag{4.17}$$

or, in terms of matrix entries, we have inductively

$$R_{i,j}(I_{E^{\otimes j}} \otimes \eta) = (I_{E^{\otimes i}} \otimes \eta) R_{i+1,j+1} \text{ for all } i,j = 0,1,\dots. \tag{4.18}$$

for $\eta = \mu^* \in (E^\sigma)^*$.

Proof. To prove that (4.17) holds for all $R \in \mathcal{F}^\infty(E,\sigma)$, it suffices to show that it holds for each R in the generating set (4.13). We are thus reduced to showing that (4.18) holds for all R of the special form $\varphi_\infty(a)$ for an $a \in \mathcal{A}$ and T_ξ for a $\xi \in E$. This in turn is a routine calculation which we leave to the reader. □

Suppose that we are given $R \in \mathcal{F}^\infty(E,\sigma)$. We regard $\mathcal{E}$ as a subspace of $\mathcal{F}^2(E,\sigma)$ via the identification $y \cong y \oplus 0 \oplus 0 \oplus \cdots$. Then the restriction of R to $\mathcal{E}$ defines an operator from $\mathcal{E}$ into $\mathcal{F}^2(E,\sigma)$ where we have a point evaluation in $\mathbb{D}((E^\sigma)^*)\times\sigma(\mathcal{A})'$ defined in (4.8). We may then define an operator $R^\wedge(\eta) \in \mathcal{L}(\mathcal{E})$ by

$$R^\wedge(\eta)e = (Re)^\wedge(\eta, I_\mathcal{E}).$$

Explicitly, we have

$$R^\wedge(\eta) = \sum_{n=0}^\infty \eta^n R_{n,0} \in \mathcal{L}(\mathcal{E}).$$

Note that, as a consequence of Proposition 4.2, the full function $\mathbf{f}(\eta,b) = (Re)^\wedge(\eta,b)$ is then determined from $R^\wedge(\eta)$ and $e \in \mathcal{E}$ according to

$$(Re)^\wedge(\eta,b) = (\iota_{\infty,\sigma}(b)Re)(\eta, I_\mathcal{E}) = (R\iota_{\infty,\sigma}(b)e)(\eta, I_\mathcal{E}) = (Rbe)(\eta) = R^\wedge(\eta)(be)$$

for $\eta \in \mathbb{D}((E^\sigma)^*)$ and $b \in \sigma(\mathcal{A})'$. This implies that if we would extend the point evaluation to $\mathbb{D}((E^\sigma)^*) \times \sigma(\mathcal{A})'$ by $R^\wedge(\eta,b)e = (Re)^\wedge(\eta,b)$, the result would just give $R^\wedge(\eta,b) = R^\wedge(\eta)b$.

It is of interest that this transform $R \to R^\wedge(\cdot)$ is multiplicative.

Proposition 4.4.

1. *Suppose that R and S are two elements of $\mathcal{F}^\infty(E,\sigma)$. Then*

   $$(RS)^\wedge(\eta) = R^\wedge(\eta)S^\wedge(\eta)$$

   for all $\eta \in \mathbb{D}((E^\sigma)^*)$.
2. *Suppose that R is an operator in $\mathcal{F}^\infty(E,\sigma)$ and that f is an element of $\mathcal{F}^2(E,\sigma)$. Then*

   $$(Rf)^\wedge(\eta,b) = R^\wedge(\eta)f^\wedge(\eta,b) \tag{4.19}$$

   for all $\eta \in \mathbb{D}((E^\sigma)^*)$ and $b \in \sigma(\mathcal{A})'$.

Proof. Suppose that $R = [R_{i,j}]_{i,j=0,1,\dots}$ is an operator in $\mathcal{F}^\infty(E,\sigma)$ and that $f = \oplus_{j=0}^\infty f_j$ is an element of $\mathcal{F}^2(E,\sigma)$. We first note that a special case of (4.18) is

$$R_{\ell,0}\eta = (I_{E^{\otimes \ell}} \otimes \eta) R_{\ell+1,1}.$$

Iteration of (4.18) in turn leads to

$$R_{\ell,0}\eta^j = I_{E^{\otimes \ell}} \otimes \eta^j R_{\ell+j,j} \colon E^{\otimes j} \otimes_\sigma \mathcal{E} \to E^{\otimes \ell} \otimes_\sigma \mathcal{E}. \tag{4.20}$$

Then we compute for $\eta \in \mathbb{D}((E^\sigma)^*)$ and $b \in \sigma(\mathcal{A})'$ that

$$\begin{aligned} R^\wedge(\eta) f^\wedge(\eta, b) &= \left(\sum_{\ell=0}^\infty \eta^\ell R_{\ell,0}\right)\left(\sum_{j=0}^\infty \eta^j (I_{E^{\otimes j}} \otimes b) f_j\right) \\ &= \sum_{\ell,j=0}^\infty \eta^\ell R_{\ell,0} \eta^j (I_{E^{\otimes j}} \otimes b) f_j \\ &= \sum_{\ell,j=0}^\infty \eta^{\ell+j} R_{\ell+j,j} (I_{E^{\otimes j}} \otimes b) f_j \text{ (by (4.20))} \\ &= \sum_{\ell,j=0}^\infty \eta^{\ell+j} (I_{E^{\otimes \ell+j}} \otimes b) R_{\ell+j,j} f_j \text{ (by (4.14))} \\ &= \sum_{n=0}^\infty \eta^n (I_{E^{\otimes n}} \otimes b) \left(\sum_{j=0}^n R_{n,j} f_j\right) \\ &= \sum_{n=0}^\infty \eta^n (I_{E^{\otimes n}} \otimes b) [Rf]_n = (Rf)^\wedge(\eta, b) \end{aligned}$$

and part (2) of the Proposition follows. Part (1) follows as the special case where $b = I_{\mathcal{E}}$ and $f = Se$ for arbitrary $e \in \mathcal{E}$. ☐

Remark 4.5. We note that a consequence of the formula (4.19) is that the operator $M_{R^\wedge}$ of multiplication by $R^\wedge$ on $H^2(E,\sigma)$

$$M_{R^\wedge} \colon f^\wedge(\eta, b) \mapsto R^\wedge(\eta) f^\wedge(\eta, b)$$

commutes with the $\sigma(\mathcal{A})'$-left action on $H^2(E,\sigma)$:

$$M_{R^\wedge}(b \cdot f^\wedge) = b \cdot M_{R^\wedge} f^\wedge \quad \text{where} \quad (b \cdot f)^\wedge(\eta', b') = f^\wedge(\eta', b'b)$$

for all $b, b' \in \sigma(A)'$ and $\eta' \in \mathbb{D}((E^\sigma)^*)$. This can also be seen as a consequence of applying the Z-transform to the identity

$$R\iota_{\infty,\sigma}(b) = \iota_{\infty,\sigma}(b) R \text{ for all } b \in \sigma(\mathcal{A})'$$

given in Proposition 4.2.

Proposition 4.4 leads immediately to the following corollary.

Corollary 4.6.

1. *The kernel of the Fourier transform* $\Phi\colon f \to f^\wedge$ *in* $\mathcal{F}^2(E,\sigma)$

$$\operatorname{Ker}\Phi = \{f \in \mathcal{F}^2(E,\sigma)\colon f^\wedge(\eta,b) = 0 \textit{ for all } \eta \in \mathbb{D}((E^\sigma)^*) \quad \textit{and} \quad b \in \sigma(\mathcal{A})'\}$$

is invariant under the analytic Toeplitz operators:

$$f^\wedge(\eta,b) = 0 \textit{ for all } \eta \in \mathbb{D}((E^\sigma)^*) \quad \textit{and} \quad b \in \sigma(\mathcal{A})',\ R \in \mathcal{F}^\infty(E,\sigma)$$
$$\Longrightarrow (Rf)^\wedge(\eta,b) = 0 \textit{ for all } \eta \in \mathbb{D}((E^\sigma)^*) \quad \textit{and} \quad b \in \sigma(\mathcal{A})'.$$

2. *The initial space* $[\mathcal{F}^2(E,\sigma)]_{\text{initial}}$ *of the Fourier transform* Φ *is invariant under the adjoints of the analytic Toeplitz operators:*

$$f \in [\mathcal{F}^2(E,\sigma)]_{\text{initial}},\ R \in \mathcal{F}^\infty(E,\sigma) \Longrightarrow R^*f \in [\mathcal{F}^2(E,\sigma)]_{\text{initial}}.$$

Explicitly, the action of R^* *on a generic vector in the spanning set* (4.12) *for* $[\mathcal{F}^2(E)\otimes_\sigma \mathcal{E}]_{\text{initial}}$ *is given by*

$$R^*(b^* \cdot k_{E,\sigma;\eta})e) = b^* \cdot k_{E,\sigma;\eta} R^\wedge(\eta)^* e.$$

Proof. If $f^\wedge(\eta,b) = 0$ for all η and b, then, by (4.19) we see immediately that

$$(Rf)^\wedge(\eta,b) = R^\wedge(\eta) f^\wedge(\eta,b) = 0$$

for all η and b as well as for any $R \in \mathcal{F}^\infty(E,\sigma)$. The first part of the second statement then follows by simply taking adjoints.

To verify the second part of the second statement, it suffices to verify on the generators $R = T_\xi$ and $R = \varphi(a)$ for $\xi \in E$ and $a \in \mathcal{A}$; this in turn is straightforward. □

Remark 4.7. We note that the definition of $R^\wedge(\eta)$ involves only the first column of R. From the relations (4.16) and (4.14) one can see that the first column of R already uniquely determines the action of R on all of $[\mathcal{F}^2(E,\sigma)]_{\text{initial}}$.

Remark 4.8. Let $\mu \in E^\sigma$ and $\eta \in (E^\sigma)^*$ and $b \in (\sigma(\mathcal{A}))'$. Then an easy verification using the relations $\mu\sigma(a) = (\varphi(a)\otimes I_\mathcal{E})\mu$ and $\sigma(a)\eta = \eta(\varphi(a)\otimes I_\mathcal{E})$ shows that

$$\eta(I_E \otimes b)\mu \in \sigma(\mathcal{A})'. \tag{4.21}$$

This observation has several consequences.

1. Given $\mu \in E^\sigma$ and $\eta \in (E^\sigma)^*$ we may define a mapping $\theta_{\eta,\mu}$ on $\sigma(\mathcal{A})'$ by

$$\theta_{\eta,\mu}(b) = \eta(I_E \otimes b)\mu.$$

Iteration of this map gives

$$\theta^2_{\eta,\mu}(b) = \eta(I_E \otimes \eta(I_E \otimes b)\mu)\mu = \eta^2(I_{E^{\otimes 2}} \otimes b)\mu^2$$

and more generally

$$\theta^n_{\eta,\mu}(b) = \eta^n(I_{E^{\otimes n}} \otimes b)\mu^n$$

where we make use of the generalized power η^n for an element η of $(E^\sigma)^*$ (and set $\mu^n = ((\mu^*)^n)^*\colon \mathcal{E} \to E^{\otimes n}\otimes_\sigma \mathcal{E}$). For $\eta, \zeta \in \mathbb{D}((E^\sigma)^*)$, we may take

$\mu = \zeta^*$ and then we have $\|\theta_{\eta,\zeta^*}\| < 1$. Then we may use the geometric series to compute the inverse of $I - \theta_{\eta,\zeta^*}$ to get

$$(I - \theta_{\eta,\zeta^*})^{-1}(b) = \sum_{n=0}^{\infty} (\theta_{\eta,\zeta^*})^n(b) = \sum_{n=0}^{\infty} \eta^n (I_{E^{\otimes n}} \otimes b)(\zeta^n)^*.$$

We conclude that the Szegö kernel (4.11) can also be written as

$$\mathbb{K}_{E,\sigma}(\eta,\zeta)[b] = (I - \theta_{\eta,\zeta^*})^{-1}(b).$$

This is the form of the Szegö kernel used in [33, 36].

2. Suppose that we are given two elements $\eta, \zeta \in E^\sigma$. The special case of (4.21) with $b = I_{\mathcal{E}}$ and $\eta = \mu'^*$ for a $\mu' \in E^\sigma$ enables us to define a $\sigma(\mathcal{A})'$-valued inner product on E^σ:

$$\langle \mu, \mu' \rangle_{E^\sigma} = \mu'^* \mu \in \sigma(\mathcal{A})' \text{ for } \mu, \mu' \in E^\sigma.$$

Moreover one can check that E^σ has a well-defined right $\sigma(\mathcal{A})'$-action

$$(\mu \cdot b)(e) = \mu(be)$$

and a well-defined left $\sigma(\mathcal{A})'$-action

$$(b \cdot \mu)(e) = (I_E \otimes b)\mu(e).$$

It is then straightforward to check that E^σ is a $\sigma(A)'$-correspondence. This observation plays a key role in the duality theory in [33] (see also Proposition 4.2 above).

Next we introduce the space

$$H^\infty(E,\sigma) = \{R^\wedge \colon R \in \mathcal{F}^\infty(E,\sigma)\},$$

where we interpret $R^\wedge$ as a function mapping $\mathbb{D}((E^\sigma)^*)$ into $\mathcal{L}(\mathcal{E})$. Then $H^\infty(E,\sigma)$ is closed under addition ($(R_1 + R_2)^\wedge = R_1^\wedge + R_2^\wedge$), scalar multiplication ($(\lambda R)^\wedge = \lambda R^\wedge$) and pointwise multiplication (Proposition 4.4 (1)). Moreover, part (2) of Proposition 4.4 implies that a function $S \in H^\infty(E,\sigma)$ defines a multiplication operator M_S on $H^2(E,\sigma)$ by

$$(M_S f^\wedge)(\eta, b) = S(\eta) f^\wedge(\eta, b) \text{ for each } \eta \in \mathbb{D}((E^\sigma)^*), b \in \sigma(\mathcal{A})', f^\wedge \in H^2(E,\sigma). \tag{4.22}$$

In fact, we have the following result.

Proposition 4.9. *A function $S : \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(\mathcal{E})$ is in $H^\infty(E,\sigma)$ if and only if S defines a multiplication operator M_S on $H^2(E,\sigma)$ by (4.22). In case $S \in H^\infty(E,\sigma)$, we have $\|M_S\| \le \|R\|$ for each $R \in \mathcal{F}^\infty(E,\sigma)$ with $S = R^\wedge$ and there exists a $R \in \mathcal{F}^\infty(E,\sigma)$ with $S = R^\wedge$ such that $\|M_S\| = \|R\|$. Moreover, if $S \in H^\infty(E,\sigma)$, then M_S is a $\sigma(\mathcal{A})$-module map that in addition commutes with the operators*

$$\Phi(I_{\mathcal{F}(E)} \otimes \mu)\Phi^* \text{ for each } \mu \in E^\sigma.$$

Here Φ is the coisometry from $\mathcal{F}^2(E,\sigma)$ into $H^2(E,\sigma)$ given by $\Phi : f \mapsto f^\wedge$.

Proof. We already observed that $S \in H^\infty(E,\sigma)$ guarantees that M_S in (4.22) defines a multiplication operator on $H^2(E,\sigma)$. Moreover, for $R \in \mathcal{F}^\infty(E,\sigma)$ with $S = R^\wedge$ we have

$$\begin{aligned}(M_S\Phi f)(\eta,b) &= (M_S f^\wedge)(\eta,b) = S(\eta)f^\wedge(\eta,b) = R^\wedge(\eta)f(\eta,b)\\ &= (Rf)^\wedge(\eta,b) = (\Phi R f)(\eta,b)\end{aligned}$$

for each $f \in \mathcal{F}^2(E,\sigma)$, $\eta \in \mathbb{D}((E^\sigma)^*)$ and $b \in \sigma(\mathcal{A})'$. Hence

$$M_S\Phi = \Phi R.$$

In particular we have $M_S = \Phi R \Phi^*$ and thus $\|M_S\| \le \|R\|$ since Φ is a coisometry.

Now assume that S defines a multiplication operator M_S on $H^2(E,\sigma)$ by (4.22). The definition of M_S and of the left action on $H^2(E,\sigma)$ in (4.10) shows that, for $b, b' \in \sigma(\mathcal{A})'$ and $\eta \in \mathbb{D}((E^\sigma)^*)$, we have

$$(M_S b' f^\wedge)(\eta,b) = S(\eta)f^\wedge(\eta,bb') = (b' M_S f^\wedge)(\eta,b) \quad \text{for each } f^\wedge \in H^2(E,\sigma).$$

Hence M_S is a $\sigma(\mathcal{A})'$-module map.

We now show that there exists $R \in \mathcal{F}^\infty(E,\sigma)$ with $R^\wedge = S$. We first note that

$$\begin{aligned}((I_{\mathcal{F}^2(E)} \otimes \mu)f)^\wedge(\eta,b) &= \sum_{n=1}^\infty \eta^n (I_{E^{\otimes n}} \otimes b)(I_{E^{\otimes n-1}} \otimes \mu) f_{n-1}\\ &= \sum_{n=1}^\infty \eta^{n-1}(I_{E^{\otimes n-1}} \otimes \eta(I_E \otimes b)\mu) f_{n-1} \qquad (4.23)\\ &= f^\wedge(\eta, \eta(I_E \otimes b)\mu) \qquad (4.24)\end{aligned}$$

where we use the observation from Remark 4.8 that $\eta(I_E \otimes b)\mu$ is in $\sigma(\mathcal{A})'$.

From (4.24), it readily follows that $I_{\mathcal{F}^2(E)} \otimes \mu$ on $\mathcal{F}^2(E,\sigma)$ leaves $\operatorname{Ker}\Phi$ invariant. The same holds for the operator $I_{\mathcal{F}^2(E)} \otimes b$. Consequently, denoting by $P (= \Phi^*\Phi)$ the projection on $\mathcal{G} = (\ker \Phi)^\perp$, we note that

$$PX = PXP \text{ for } X = I_{\mathcal{F}^2(E)} \otimes \mu, I_{\mathcal{F}^2(E)} \otimes b.$$

We show that the operator $\Phi^* M_S \Phi$ commutes with $I_{\mathcal{F}(E)} \otimes b'$ for all $b' \in (\sigma(\mathcal{A}))'$. To see this, let $f \in \mathcal{F}^2(E,\sigma)$ and $\Phi^* M_S \Phi (I_{\mathcal{F}(E)} \otimes b')f = g$. Due to (4.24), we have $g^\wedge(\eta,b) = S(\eta,bb')$. Now if we let $\Phi^* M_S \Phi f = h$, it follows that $h^\wedge(\eta,b) = S(\eta)f^\wedge(\eta,b)$ and consequently,

$$((I_{\mathcal{F}(E)} \otimes b')\Phi^* M_S \Phi f)^\wedge(\eta,b) = S(\eta)\hat{f}(\eta,bb')$$

and the claim follows. A similar computation using (4.24) shows that $P(I_{\mathcal{F}(E)} \otimes \mu)A = AP(I_{\mathcal{F}(E)} \otimes \mu)|\mathcal{G}$ for all $\mu \in E^\sigma$, where $A = \Phi^* M_S \Phi|\mathcal{G}$.

We recall now that the maps $\mu \in E^\sigma, b \in \sigma(\mathcal{A})'$ form an isometric covariant representation of the $\sigma(\mathcal{A})'$-correspondence E^σ (see pages 369-370 in [33] – the precise definition is *covariant representation* is given in the text surrounding formulas (4.26)–(4.28) below). We may now apply the commutant lifting theorem for covariant representations of a correspondence due to Muhly-Solel (see Theorem 4.4, [31]) to obtain an operator that commutes with the operators $I_{\mathcal{F}^2(E)} \otimes \mu$ and

$I_{\mathcal{F}^2(E)} \otimes b$ (which implies $R \in \mathcal{F}^\infty(E)$ by Proposition 4.2) which moreover satisfies $PR = AP$. This immediately implies that $R^\wedge = S$. Furthermore, we can choose R such that $\|R\| = \|M_S\|$. □

We note that any $R \in \mathcal{F}^\infty(E, \sigma)$ is of the form $\widetilde{R} \otimes I_\mathcal{E}$ for a $\widetilde{R} \in \mathcal{F}^\infty(E)$. Moreover, the map $\widetilde{R} \mapsto R = \widetilde{R} \otimes I_\mathcal{E}$ is an $\mathcal{L}(\mathcal{F}^2(E, \sigma))$-valued representation of $\mathcal{F}^\infty(E)$ which actually extends to a $*$-representation $T \mapsto T \otimes I_\mathcal{E}$ of all of $\mathcal{L}^a(\mathcal{F}^2(E))$ – the restriction of $T \mapsto T \otimes I_\mathcal{E}$ to $T \in \mathcal{F}^\infty(E)$ is called the *induced representation* of $\mathcal{F}^\infty(E)$ in the terminology of [31, 33]. The content of Proposition 4.4 is that, for each $\eta \in \mathbb{D}((E^\sigma)^*)$, the map $R \mapsto R^\wedge(\eta)$ is an $\mathcal{L}(\mathcal{E})$-valued representation of $\mathcal{F}^\infty(E, \sigma)$. It follows that the composition

$$\pi_\eta(\widetilde{R}) = (\widetilde{R} \otimes I_\mathcal{E})^\wedge(\eta) \tag{4.25}$$

is an (even completely contractive) representation of $\mathcal{F}^\infty(E)$ (see [33]). For *some* $\eta \in (E^\sigma)^*$ of norm equal to 1, π_η still defines a representation of $\mathcal{F}(E)$. It is the case that every η in the closed unit ball of $(E^\sigma)^*$ gives rise to a completely contractive representation of $\mathcal{T}_+(E)$ (the norm-closure of the span of left multipliers $\varphi_\infty(a)$ $(a \in \mathcal{A})$ and creation operators T_ξ $(\xi \in E)$ in $\mathcal{L}^a(\mathcal{F}^2(E))$), while it is not clear for which such η the representation can be extended to $\mathcal{F}^\infty(E)$ – this is one of the open problems in the theory (see [33]). It is the case that each completely contractive representation π of $\mathcal{F}^\infty(E)$ comes from an $\eta \in \overline{\mathbb{D}}((E^\sigma)^*)$ for some weak-$*$ continuous $*$-representation $\sigma\colon \mathcal{A} \to \mathcal{L}(\mathcal{E})$. Indeed, given a completely contractive representation $\pi\colon \mathcal{F}^\infty(E) \to \mathcal{L}(\mathcal{E})$, one can construct σ and η as follows. Define $\sigma\colon \mathcal{A} \to \mathcal{L}(\mathcal{E})$ by

$$\sigma(a) = \pi(\varphi_\infty(a)). \tag{4.26}$$

Then define $\boldsymbol{\eta}\colon E \to \mathcal{L}(\mathcal{E})$ by

$$\boldsymbol{\eta}(\xi) = \pi(T_\xi). \tag{4.27}$$

We wish to verify that

$$\boldsymbol{\eta}(\varphi(a)\xi \cdot a') = \sigma(a)\boldsymbol{\eta}(\xi)\sigma(a'), \tag{4.28}$$

i.e., that the pair $(\boldsymbol{\eta}, \sigma)$ is a *covariant representation* of E in the terminology of Muhly-Solel [31, 33]. As a first step for the verification of (4.28), one can easily check that

$$T_{\varphi(a)\xi \cdot a'} = \varphi_\infty(a) T_\xi \varphi_\infty(a') \text{ for } a \in \mathcal{A},\ \xi \in E.$$

We then compute

$$\begin{aligned} \boldsymbol{\eta}(\varphi(a)\xi \cdot a') &= \pi(T_{\varphi(a)\xi \cdot a'}) \\ &= \pi(\varphi_\infty(a) T_\xi \varphi_\infty(a')) \\ &= \pi(\varphi_\infty(a))\pi(T_\xi)\pi(\varphi_\infty(a')) \\ &= \sigma(a)\boldsymbol{\eta}(\xi)\sigma(a') \end{aligned}$$

and (4.28) follows. As in [31], a covariant representation $(\boldsymbol{\eta}, \sigma)$ of E determines an element $\eta\colon E \otimes_\sigma \mathcal{E} \to \mathcal{E}$ of $(E^\sigma)^*$ according to the formula

$$\eta(\xi \otimes e) = \boldsymbol{\eta}(\xi)e. \tag{4.29}$$

Here note that the property $\boldsymbol{\eta}(\xi \cdot a') = \boldsymbol{\eta}(\xi)\sigma(a')$ is what is needed to verify that (4.29) is well defined while the property $\boldsymbol{\eta}(\varphi(a)\xi) = \sigma(a)\boldsymbol{\eta}(\xi)$ is what is needed to verify that η is in $(E^\sigma)^*$, i.e., that η has the $\mathcal{A}$-module-map property

$$\eta(\varphi(a) \otimes I_{\mathcal{E}}) = \sigma(a)\eta.$$

There is a converse: given an element $\eta \in \mathbb{D}((E^\sigma)^*)$, we may use (4.29) to define $\boldsymbol{\eta}$ so that $(\boldsymbol{\eta}, \sigma)$ is a completely contractive covariant representation of E. The mapping π given in (4.26)–(4.27) then extends to a representation of $\mathcal{F}^\infty(E)$ (see [31]). For our situation here where $(\boldsymbol{\eta}, \sigma)$ is given in terms of a representation π via (4.26)–(4.27), we see that a representation π of $\mathcal{F}^\infty(E)$ determines a representation $\sigma = \sigma_\pi$ of $\mathcal{A}$ according to (4.26) along with an element η_π of $(E^\sigma)^*$ according to the formula

$$\eta_\pi(\xi \otimes e) = \pi(T_\xi)e.$$

It is then straightforward to check that the formula

$$\pi(\widetilde{R}) = (\widetilde{R} \otimes I_{\mathcal{E}})^\wedge(\eta_\pi) \tag{4.30}$$

holds for the cases where

$$\widetilde{R} = \varphi_\infty(a) \text{ for some } a \in \mathcal{A}, \qquad \widetilde{R} = T_\xi \text{ for some } \xi \in E.$$

Under the assumption that π is continuous with respect to the weak-$*$ topologies on $\mathcal{F}^\infty(E)$ and $\mathcal{L}(\mathcal{E})$, it then follows that (4.30) holds for all $\widetilde{R} \in \mathcal{F}^\infty(E)$, i.e., we recover π as $\pi = \pi_{\eta_\pi}$ where in general π_η is given by (4.25).

It is of interest to apply this construction to the induced representation

$$\pi_{\text{ind}} \colon \widetilde{R} \mapsto \widetilde{R} \otimes I_{\mathcal{E}} \tag{4.31}$$

of $\mathcal{F}^\infty(E)$ into $\mathcal{L}(\mathcal{F}^2(E, \sigma))$. We collect this result in the following proposition.

Proposition 4.10. *Suppose that we are given an $\mathcal{A}$-correspondence E together with a representation $\sigma \colon \mathcal{A} \to \mathcal{L}(\mathcal{E})$ for a Hilbert space $\mathcal{E}$. Let $\pi_{\text{ind}} \colon \mathcal{F}^\infty(E) \to \mathcal{L}(\mathcal{F}^2(E,\sigma))$ be the induced representation as in* (4.31). *Define $\boldsymbol{\eta}_{\text{ind}} \colon E \to \mathcal{L}(\mathcal{F}^2(E, \sigma))$ and $\sigma_{\text{ind}} \colon \mathcal{A} \to \mathcal{L}(\mathcal{F}^2(E, \sigma))$ by*

$$\boldsymbol{\eta}_{\text{ind}}(\xi) = T_{\xi,\sigma}, \qquad \sigma_{\text{ind}}(a) = \varphi_{\infty,\sigma}(a).$$

Then $(\boldsymbol{\eta}_{\text{ind}}, \sigma_{\text{ind}})$ is an (isometric) covariant representation of E with element $\eta_{\text{ind}} \colon E \otimes \mathcal{F}^2(E, \sigma) \to \mathcal{F}^2(E, \sigma)$ of $(E^{\sigma_{\text{ind}}})^*$ associated with $(\boldsymbol{\eta}_{\text{ind}}, \sigma_{\text{ind}})$ as in (4.29) *given by*

$$\eta_{\text{ind}} \colon \xi \otimes \left[\oplus_{n=0}^\infty \xi^{(n)} \otimes e_n\right] \mapsto 0 \oplus \left[\oplus_{n=1}^\infty \xi \otimes \xi^{(n-1)} \otimes e_{n-1}\right].$$

Moreover, we recover $R = \widetilde{R} \otimes I_{\mathcal{E}} \in \mathcal{F}^\infty(E, \sigma)$ via the point evaluation

$$\widetilde{R} \otimes I_{\mathcal{E}} = (\widetilde{R} \otimes I_{\mathcal{F}^2(E,\sigma)})^\wedge(\eta_{\text{ind}}).$$

Proof. The proof is a simple specialization of the general construction sketched in the paragraph preceding the statement of the proposition. □

It will be convenient to work also with the analytic Toeplitz operators acting between $H^2(E,\sigma)$-spaces of different multiplicity. For this purpose, we suppose that $\mathcal{U}$ and $\mathcal{Y}$ are two additional auxiliary Hilbert spaces (to be thought of as an *input space* and *output space* respectively). We consider higher multiplicity versions of $H^2(E,\sigma)$ by tensoring with an auxiliary Hilbert space (which is to be thought of as adding multiplicity):

$$H^2_{\mathcal{U}}(E,\sigma) := H^2(E,\sigma) \otimes_{\mathbb{C}} \mathcal{U}, \qquad H^2_{\mathcal{Y}}(E,\sigma) := H^2(E,\sigma) \otimes_{\mathbb{C}} \mathcal{Y}.$$

Here we view $\mathcal{U}$ and $\mathcal{Y}$ as $(\mathbb{C},\mathbb{C})$-correspondences and apply the tensor product construction of Definition 3.1 (2). The space $H^2_{\mathcal{U}}(E,\sigma)$ then is a reproducing kernel $(\sigma(\mathcal{A})', \mathcal{L}(\mathcal{E}\otimes\mathcal{U}))$-correspondence on $\mathbb{D}((E^\sigma)^*)$ where the point evaluation at a point $(\eta,b) \in \mathbb{D}((E^\sigma)^*) \times \sigma(\mathcal{A})'$ of a function $f^\wedge \otimes u \in H^2_{\mathcal{U}}(E,\sigma)$ (with $f^\wedge \in H^2(E,\sigma)$ and $u \in \mathcal{U}$) is given by $(f^\wedge \otimes u)(\eta,b) = f^\wedge(\eta,b)\otimes u \in \mathcal{E}\otimes\mathcal{U}$. Moreover, note that the left $\sigma(\mathcal{A})'$-action is given by $b \mapsto b\otimes I_{\mathcal{U}}$. The completely positive kernel $\mathbb{K}_{(E,\sigma)\otimes\mathcal{U}}$ associated with it as in Theorem 3.6 is given by

$$\mathbb{K}_{(E,\sigma)\otimes\mathcal{U}}(\eta,\zeta)[b] = \mathbb{K}_{E,\sigma}(\eta,\zeta)[b]\otimes I_{\mathcal{U}},$$

where $\mathbb{K}_{E,\sigma}$ denotes the kernel for $H^2(E,\sigma)$ defined in Theorem 4.1. Similar statements hold for $H^2_{\mathcal{Y}}(E,\sigma)$, where the analogous kernel is denoted by $\mathbb{K}_{(E,\sigma)\otimes\mathcal{Y}}$.

We now define a higher-multiplicity version of the algebra of analytic Toeplitz operators $H^\infty(E,\sigma)$ to be the linear space

$$H^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma) := H^\infty(E,\sigma)\otimes\mathcal{L}(\mathcal{U},\mathcal{Y}).$$

This space consists of $\mathcal{L}(\mathcal{E}\otimes\mathcal{U},\mathcal{E}\otimes\mathcal{Y})$-valued functions on $\mathbb{D}((E^\sigma)^*)$, with point evaluation of an element $S\otimes N \in H^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma) = H^\infty(E,\sigma)\otimes\mathcal{L}(\mathcal{U},\mathcal{Y})$ at $\eta \in \mathbb{D}((E^\sigma)^*)$ given by $(S\otimes N)(\eta) = S(\eta)\otimes N$. Moreover, the functions in $H^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ define multiplication operators in $\mathcal{L}(H^2_{\mathcal{U}}(E,\sigma), H^2_{\mathcal{Y}}(E,\sigma))$, in the same way as $H^\infty(E,\sigma)$. For $S\otimes N \in H^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$, $S\in H^\infty(E,\sigma)$ and $N\in\mathcal{L}(\mathcal{U},\mathcal{Y})$, the multiplication operator $M_{S\otimes N}$ becomes $M_{S\otimes N} = M_S\otimes N$.

In addition there are Fock space versions of all these spaces, namely

$$\mathcal{F}^2_{\mathcal{U}}(E,\sigma) := \mathcal{F}^2(E,\sigma)\otimes\mathcal{U}, \qquad \mathcal{F}^2_{\mathcal{Y}}(E,\sigma) := \mathcal{F}^2(E,\sigma)\otimes\mathcal{Y},$$
$$\mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma) = \mathcal{F}^\infty(E,\sigma)\otimes\mathcal{L}(\mathcal{U},\mathcal{Y}).$$

Point evaluation for elements in $\mathcal{F}^2_{\mathcal{U}}(E,\sigma)$ and points in $\mathbb{D}((E^\sigma)^*)\times\sigma(\mathcal{A})'$ (and similarly for elements in $\mathcal{F}^2_{\mathcal{Y}}(E,\sigma)$) is determined by attaching to $f\otimes u \in \mathcal{F}^2_{\mathcal{U}}(E,\sigma)$, $f\in\mathcal{F}^2(E,\sigma)$ and $u\in\mathcal{U}$, and $(\eta,b)\in\mathbb{D}((E^\sigma)^*)\times\sigma(\mathcal{A})'$ the value $(f\otimes u)^\wedge(\eta,b) = f^\wedge(\eta,b)\otimes u$, so that the map

$$\Phi_{\mathcal{U}} : f_u \mapsto f_u^\wedge \text{ for } f_u \in \mathcal{F}^2_{\mathcal{U}}(E,\sigma)$$

defines a coisometry from $\mathcal{F}^2_{\mathcal{U}}(E,\sigma)$ onto $H^2_{\mathcal{U}}(E,\sigma)$. The analogous coisometry for $\mathcal{F}^2_{\mathcal{Y}}(E,\sigma)$ is denoted by $\Phi_{\mathcal{Y}}$. Similarly we determine point evaluation for elements in $\mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ and points in $\mathbb{D}((E^\sigma)^*)$ by attaching to $R\otimes X \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$, $R\in\mathcal{F}^\infty(E,\sigma)$ and $X\in\mathcal{L}(\mathcal{U},\mathcal{Y})$, and $\eta\in\mathbb{D}((E^\sigma)^*)$ the value

$$(R\otimes X)^\wedge(\eta) := R^\wedge(\eta)\otimes X \in \mathcal{L}(\mathcal{E}\otimes\mathcal{U},\mathcal{E}\otimes\mathcal{Y}). \tag{4.32}$$

Then $H^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ is recovered as

$$\{R^\wedge\colon R \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)\},$$

where $R^\wedge$ should be interpreted as a function mapping $\mathbb{D}((E^\sigma)^*)$ into $\mathcal{L}(\mathcal{E}\otimes\mathcal{U},\mathcal{E}\otimes\mathcal{Y})$, while the space of multiplication operators in $\mathcal{L}(H^2_{\mathcal{U}}(E,\sigma),H^2_{\mathcal{Y}}(E,\sigma))$ defined by $H^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ is given by

$$\{\Phi_{\mathcal{Y}} R \Phi^*_{\mathcal{U}}\colon R \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)\}.$$

In fact, it is easy to check that Proposition 4.9 implies that for $S \in H^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ we have $M_S = \Phi_{\mathcal{Y}} R \Phi^*_{\mathcal{U}}$ whenever $R \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ satisfies $S = R^\wedge$, so that $\|M_S\| \leq \|R\|$, and that there exists a $R \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ with $S = R^\wedge$ and $\|M_S\| = \|R\|$.

Alternatively, $\mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ can be characterized as bounded operators from $\mathcal{F}^2_{\mathcal{U}}(E,\sigma)$ to $\mathcal{F}^2_{\mathcal{Y}}(E,\sigma)$ with block-matrix representation

$$R = [R_{i,j}]_{i,j=0,1,\dots} \text{ with } R_{i,j}\colon E^{\otimes j} \otimes_\sigma \mathcal{E} \otimes \mathcal{U} \to E^{\otimes i} \otimes_\sigma \mathcal{E} \otimes \mathcal{Y}$$

subject to

$$R_{i,j}(I_{E^{\otimes j}} \otimes b \otimes I_{\mathcal{U}}) = (I_{E^{\otimes i}} \otimes b \otimes I_{\mathcal{Y}})R_{i,j} \text{ for } b \in \sigma(\mathcal{A})',$$
$$R_{i+1,j+1}(I_{E^{\otimes j}} \otimes \eta^* \otimes I_{\mathcal{U}}) = (I_{E^{\otimes i}} \otimes \eta^* \otimes I_{\mathcal{Y}})R_{i,j} \text{ for } \eta^* \in E^\sigma.$$

For such $R \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ point evaluation in $\eta \in \mathbb{D}((E^\sigma)^*)$ can be written as

$$R^\wedge(\eta) = \sum_{n=0}^\infty (\eta^n \otimes I_{\mathcal{Y}})R_{n,0}.$$

In addition it is routine to see that part (1) of Proposition 4.4 can be extended to the following statement: *if $S \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ and $R \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{Y},\mathcal{Z})}(E,\sigma)$, then $RS \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Z})}(E,\sigma)$ and*

$$(RS)^\wedge(\eta) = R^\wedge(\eta)S^\wedge(\eta). \tag{4.33}$$

Remark 4.11. Suppose that $R \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ has the form

$$R = \widetilde{R} \otimes_\sigma I_{\mathcal{E}} \otimes X \tag{4.34}$$

where $\widetilde{R} \in \mathcal{F}^\infty(E)$ and $X \in \mathcal{L}(\mathcal{U},\mathcal{Y})$. In particular the point evaluation (4.32) defines $R^\wedge(\eta) \in \mathcal{L}(\mathcal{E}\otimes\mathcal{U},\mathcal{E}\otimes\mathcal{Y})$ for each $\eta \in \mathbb{D}((E^\sigma)^*)$. Suppose now that $\sigma'\colon \mathcal{A} \to \mathcal{L}(\mathcal{E}')$ is another $*$-representation of $\mathcal{A}$ and $\eta' \in \mathbb{D}((E^{\sigma'})^*)$. Then we may define a related function $\eta' \mapsto R^{\wedge\prime}(\eta') \in \mathcal{L}(\mathcal{E}'\otimes\mathcal{U},\mathcal{E}'\otimes\mathcal{Y})$ by

$$R^{\wedge\prime}(\eta') = (\widetilde{R} \otimes_{\sigma'} I_{\mathcal{E}'} \otimes X)^\wedge(\eta').$$

While not all elements R of $\mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ are of the special form (4.34), finite linear combinations of elements of the special form (4.34) are weak-$*$ dense in $\mathcal{F}_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$. By using linearity and a limiting process, one can then make sense of $R^{\wedge\prime}(\eta') \in \mathcal{L}(\mathcal{E}'\otimes\mathcal{U},\mathcal{E}'\otimes\mathcal{Y})$ for any $\eta' \in \mathbb{D}((E^{\sigma'})^*)$. This fact will be useful for the formulation of condition $(1'')$ in Theorem 5.1 below.

5. The Schur class associated with (E, σ)

Given a correspondence-representation pair (E, σ) (where $\sigma\colon \mathcal{A} \to \mathcal{L}(\mathcal{E})$) along with two auxiliary Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, we define the associated Schur class $\mathcal{S}_{E,\sigma}(\mathcal{U}, \mathcal{Y})$ by

$$\begin{aligned}\mathcal{S}_{E,\sigma}(\mathcal{U}, \mathcal{Y}) = \{ & S\colon \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(\mathcal{E} \otimes \mathcal{U}, \mathcal{E} \otimes \mathcal{Y})\colon\ S(\eta) = R^\wedge(\eta),\ \text{for all} \\ & \eta \in \mathbb{D}((E^\sigma)^*), \text{for some } R \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E, \sigma) \text{ with } \|R\| \le 1\}.\end{aligned} \tag{5.1}$$

We have the following characterization of the Schur class $\mathcal{S}_{E,\sigma}(\mathcal{U}, \mathcal{Y})$ analogous to the characterization of the classical Schur class given in Theorem 1.1 and to the multivariable extensions in Theorems 2.1 and 2.3. When specialized to the classical case (see Section 6.1 below), (5.1) gives the classical Schur class as defined in the Introduction, but from a different point of view. Rather than simply holomorphic, contractive, $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function on the unit disk, (5.1) asks us to think of such functions as analytic functions $F(z) \sim \sum_{n=0}^\infty F_n z^n$ on $\mathbb{D}$ whose Taylor coefficients $\{F_n\}_{n\in\mathbb{Z}_+}$ induce a Toeplitz matrix

$$T_F = \begin{bmatrix} F_0 & 0 & 0 & \dots \\ F_1 & F_0 & 0 & \dots \\ F_2 & F_1 & F_0 & \\ \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

which defines a contraction operator from $\ell^2_{\mathcal{U}}(\mathbb{Z}_+)$ to $\ell^2_{\mathcal{Y}}(\mathbb{Z}_+)$. Thus the label (1) in Theorem 5.1, when specialized to the classical case, corresponds to a somewhat different statement than (1) in Theorem 1.1. The other labels (1′), (1″), (2) and (3) in Theorem 5.1 correspond exactly to the corresponding statements in Theorems 1.1, 2.1 and 2.3.

Theorem 5.1. *Suppose that we are given a correspondence-representation pair (E, σ) (where $\sigma\colon \mathcal{A} \to \mathcal{L}(\mathcal{E})$) along with auxiliary Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ and an operator-valued function $S\colon \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(\mathcal{E} \otimes \mathcal{U}, \mathcal{E} \otimes \mathcal{Y})$. Then the following conditions are equivalent:*

(1) *$S \in \mathcal{S}_{E,\sigma}(\mathcal{U}, \mathcal{Y})$, i.e., there exists an $R \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E, \sigma)$ with $\|R\| \le 1$ such that $S(\eta) = R^\wedge(\eta)$ for all $\eta \in \mathbb{D}((E^\sigma)^*)$.*

(1′) *The multiplication operator*

$$M_S\colon f(\eta, b) \mapsto S(\eta) f(\eta, b)$$

maps $H^2_{\mathcal{U}}(E, \sigma)$ contractively into $H^2_{\mathcal{Y}}(E, \sigma)$.

(1″) *S is such that $S(\eta) = R^\wedge(\eta)$ for all $\eta \in \mathbb{D}((E^\sigma)^*)$ for an $R \in \mathcal{F}^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E, \sigma)$ with the additional property: for each representation $\sigma'\colon \mathcal{A} \to \mathcal{L}(\mathcal{E}')$ and $\eta' \in \mathbb{D}((E^{\sigma'})^*)$ it happens that*

$$\|R^{\wedge'}(\eta')\| \le 1,$$

where $R^{\wedge'}(\eta')$ is defined as in Remark 4.11.

(2) *The function* $\mathbb{K}_S : \mathbb{D}((E^\sigma)^*) \times \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(\sigma(\mathcal{A})', \mathcal{L}(\mathcal{E} \otimes \mathcal{Y}))$ *defined by*

$$\mathbb{K}_S(\eta, \zeta)[b] := \mathbb{K}_{(E,\sigma)\otimes\mathcal{Y}}(\eta, \zeta)[b] - S(\eta)\mathbb{K}_{(E,\sigma)\otimes\mathcal{U}}(\eta, \zeta)[b]S(\zeta)^*$$

is completely positive, or more explicitly, there exists an auxiliary Hilbert space $\mathcal{H}$*, an operator-valued function* $H\colon \mathbb{D}((E^\sigma)^*) \to \mathcal{L}(\mathcal{H}, \mathcal{E} \otimes \mathcal{Y})$ *and a* $*$*-representation* π *of* $\sigma(\mathcal{A})'$ *on* $\mathcal{H}$ *so that*

$$\left(\sum_{n=0}^{\infty} \eta^n (I_{E^{\otimes n}} \otimes b)(\zeta^n)^*\right) \otimes I_{\mathcal{Y}} - S(\eta) \left[\left(\sum_{n=0}^{\infty} \eta^n (I_{E^{\otimes n}} \otimes b)(\zeta^n)^*\right) \otimes I_{\mathcal{U}}\right] S(\zeta)^* = H(\eta)\pi(b)H(\zeta)^* \tag{5.2}$$

for all $\eta, \zeta \in \mathbb{D}((E^\sigma)^*)$ *and* $b \in \sigma(\mathcal{A})'$.

(3) *There exists an auxiliary Hilbert space* $\mathcal{H}$*, a* $*$*-representation* $\pi\colon \sigma(\mathcal{A})' \to \mathcal{L}(\mathcal{H})$*, and a coisometric colligation*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \otimes \mathcal{U} \end{bmatrix} \to \begin{bmatrix} E^\sigma \otimes \mathcal{H} \\ \mathcal{E} \otimes \mathcal{Y} \end{bmatrix} \tag{5.3}$$

which is a $\sigma(\mathcal{A})'$*-module map, i.e.,*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \pi(b)h \\ (b \otimes I_{\mathcal{U}})\mathbf{u} \end{bmatrix} = \begin{bmatrix} (I_E \otimes b) \otimes I_{\mathcal{H}} & 0 \\ 0 & b \otimes I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} h \\ \mathbf{u} \end{bmatrix} \tag{5.4}$$

for $h \in \mathcal{H}$ *and* $\mathbf{u} \in \mathcal{E} \otimes \mathcal{U}$*, so that* S *can be realized as*

$$S(\eta) = D + C(I - L_{\eta^*}^* A)^{-1} L_{\eta^*}^* B. \tag{5.5}$$

Here $L_{\eta^*} : \mathcal{H} \to E^\sigma \otimes \mathcal{H}$ *is given by*

$$L_{\eta^*} h = \eta^* \otimes h \text{ for each } h \in \mathcal{H}.$$

Proof. Both **(1)** $\implies$ **(1′)** and **(1′)** $\implies$ **(1)** follow immediately after extending Proposition 4.9 to the case with the added multiplicity as mentioned at the end of Section 4.

(1) $\implies$ **(1″):** Given $\eta' \in \mathbb{D}((E^{\sigma'})^*)$ (so $\|\eta'\| < 1$), by the dilation result in [33, Theorem 2.13] (see also [32]) we know that η' has a dilation to an induced representation $\eta_{\text{ind}}\colon \mathcal{F}^\infty(E) \to \mathcal{L}(\mathcal{F}^2(E, \sigma_{\text{ind}}))$ associated with a representation $\sigma_{\text{ind}}\colon \mathcal{A} \to \mathcal{L}(\mathcal{E}_{\text{ind}})$. As R is contractive by assumption, it then follows that $R^{\wedge\text{ind}}(\eta_{\text{ind}})$ is also contractive. Since η_{ind} is a dilation of η, we then also have

$$\|R^{\wedge\prime}(\eta')\| = \|P_{\mathcal{E}\otimes\mathcal{Y}} R^{\wedge\text{ind}}(\eta_{\text{ind}})|_{\mathcal{E}\otimes\mathcal{U}}\| \le \|R^{\wedge\text{ind}}(\eta_{\text{ind}})\| = \|R\| \le 1$$

and (1″) follows.

(1″) $\implies$ **(2):** This implication requires an adaptation of the GNS/HB construction to the setting of completely positive (rather than classical positive) kernels. If $\mathbb{K}(\omega', \omega)[a]$ is a completely positive kernel, then

$$K((\omega', a'), (\omega, a)) = \mathbb{K}(\omega, \omega')[a^* a']$$

is a positive kernel in the classical sense on $\Omega \times \mathcal{A}$. In this way one can reduce to the classical setting and adapt the GNS/HB construction in [23] to the situation here. We leave complete details for another occasion.

(1″) ⟹ (1′): Assume that $R \in \mathcal{F}^{\infty}_{\mathcal{L}(\mathcal{U},\mathcal{Y})}(E,\sigma)$ and that $S = R^{\wedge}$. From Proposition 4.10 extended to the higher multiplicity setting, we see that we recover R via the point-evaluation calculus as

$$R = R^{\wedge}(\eta_{\rm ind}).$$

Hence we also recover R as the strong limit

$$R = \lim_{r \uparrow 1} R^{\wedge}(r\eta_{\rm ind}).$$

The assumption (1″) tells us that

$$\|R^{\wedge}(r\eta_{\rm ind})\| \le 1$$

for each $r < 1$. Hence $\|R\| \le 1$.

(1′) ⟹ (2) Assume that M_S is as in (1′). From the definitions we see that

$$(b \cdot M_S f^{\wedge})(\eta', b') = S(\eta') f^{\wedge}(\eta', b'b) = (M_S(b \cdot f^{\wedge}))(\eta', b')$$

and hence any multiplication operator M_S is a $\sigma(\mathcal{A})'$-module map. The computation

$$\begin{aligned}
\langle M_S f, b \cdot (k_{E,\sigma;\zeta} \otimes I_{\mathcal{Y}})(e \otimes y)\rangle_{H^2_{\mathcal{Y}}(E,\sigma)} &= \langle b^* \cdot M_S f, (k_{E,\sigma;\zeta} \otimes I_{\mathcal{Y}})(e \otimes y)\rangle_{H^2_{\mathcal{Y}}(E,\sigma)} \\
&= \langle M_S(b^* \cdot f), (k_{E,\sigma;\zeta} \otimes I_{\mathcal{Y}})(e \otimes y)\rangle_{H^2_{\mathcal{Y}}(E,\sigma)} \\
&= \langle S(\zeta)(b^* \cdot f)(\zeta), e \otimes y\rangle_{\mathcal{E}\otimes\mathcal{Y}} \\
&= \langle b^* \cdot f, (k_{E,\sigma;\zeta} \otimes I_{\mathcal{U}}) S(\zeta)^*(e \otimes y)\rangle_{H^2_{\mathcal{U}}(E,\sigma)} \\
&= \langle f, b \cdot (k_{E,\sigma;\zeta} \otimes I_{\mathcal{U}}) S(\zeta)^*(e \otimes y)\rangle_{H_{\mathcal{U}}(E,\sigma)}
\end{aligned}$$

shows that

$$M_S^* \colon b \cdot (k_{E,\sigma;\zeta} \otimes I_{\mathcal{Y}})(e \otimes y) \mapsto b \cdot (k_{E,\sigma;\zeta} \otimes I_{\mathcal{U}}) S(\zeta)^*(e \otimes y). \tag{5.6}$$

Since $\|M_S\| \le 1$ by assumption, for any finite collection of $b_j \in \sigma(\mathcal{A}')$, $\zeta_j \in \mathbb{D}((E^\sigma)^*)$ and $e_j \otimes y_j \in \mathcal{E} \otimes \mathcal{Y}$ $(j = 1, \dots, N)$, we have

$$\left\| \sum_{j=1}^n b_j \cdot (k_{E,\sigma;\zeta_j} \otimes I_{\mathcal{Y}})(e_j \otimes y_j) \right\|^2 - \left\| M_S^* \sum_{j=1}^n b_j \cdot (k_{E,\sigma;\zeta_j} \otimes I_{\mathcal{Y}})(e_j \otimes y_j) \right\|^2 \ge 0. \tag{5.7}$$

Expanding out inner products and using (5.6) and the basic general identities

$$\begin{aligned}
&\langle b' \cdot (k_{e,\sigma;\zeta} \otimes I_{\mathcal{Y}})(e' \otimes y'), b \cdot (k_{E,\sigma;\eta} \otimes I_{\mathcal{Y}})(e \otimes y)\rangle_{H^2_{\mathcal{Y}}(E,\sigma)} \\
&\qquad = \langle \mathbb{K}_{(E,\sigma)\otimes\mathcal{Y}}(\eta,\zeta)[b^* b'](e' \otimes y'), e \otimes y\rangle_{\mathcal{E}\otimes\mathcal{Y}}, \\
&\langle b' \cdot (k_{E,\sigma,\zeta} \otimes I_{\mathcal{U}}) S(\zeta)^*(e' \otimes y'), b \cdot (k_{E,\sigma,\eta} \otimes I_{\mathcal{U}}) S(\eta)^*(e \otimes y)\rangle_{H^2_{\mathcal{U}}(E)} \\
&\qquad = \langle S(\eta)\mathbb{K}_{(E,\sigma)\otimes\mathcal{U}}(\eta,\zeta)[b^* b'] S(\zeta)^*(e' \otimes y'), e \otimes y\rangle_{\mathcal{E}\times\mathcal{Y}}
\end{aligned}$$

we see that the left-hand side of (5.7) is equal to

$$\sum_{i,j=1}^{N} \langle \mathbb{K}_S(\zeta_i, \zeta_j)[b_i^* b_j](e_j \otimes y_j),\, e_i \otimes y_i \rangle_{\mathcal{E}\otimes\mathcal{Y}}$$

and we conclude that $\mathbb{K}_S$ is a completely positive kernel as wanted. The characterization given in (5.2) follows from part (2) of Theorem 3.6.

(2) $\Longrightarrow$ (3): The argument here is an adaptation of the proof of Theorem 3.5 in [36] to our setting. Assume that (2) holds. By Remark 4.8, the equality (5.2) can be rewritten as

$$(I - \theta_{\eta,\zeta^*})^{-1}(b) \otimes I_{\mathcal{Y}} - S(\eta)[(I - \theta_{\eta,\zeta^*})^{-1}(b) \otimes I_{\mathcal{U}}]S(\zeta)^* = H(\eta)\pi(b)H(\zeta)^*.$$

Replace b by $[I - \theta_{\eta,\zeta^*}](b) = b - \theta_{\eta,\zeta^*}(b)$ to rewrite this last expression as an *Agler decomposition* (see [1])

$$b \otimes I_{\mathcal{Y}} - S(\eta)(b \otimes I_{\mathcal{U}})S(\zeta)^* = H(\eta)\pi\left(b - \eta(I_E \otimes b)\zeta^*\right)H(\zeta)^*. \tag{5.8}$$

Rearranging and conjugating by two generic vectors $\mathbf{y}$ and $\mathbf{y}'$ in $\mathcal{E} \otimes \mathcal{Y}$ then gives us

$$\begin{aligned} \mathbf{y}^* H(\eta)\pi(b)H(\zeta)^*\mathbf{y}' &+ \mathbf{y}^* S(\eta)(b \otimes I_{\mathcal{U}})S(\zeta)^*\mathbf{y}' \\ &= \mathbf{y}^* H(\eta)\pi(\eta(I_E \otimes b)\zeta^*)H(\zeta)^*\mathbf{y}' + \mathbf{y}^*(b \otimes I_{\mathcal{Y}})\mathbf{y}'. \end{aligned} \tag{5.9}$$

From Remark 4.8 we know that E^σ is a $\sigma(\mathcal{A})'$-correspondence. We may also view the Hilbert space $\mathcal{H}$ as a $(\sigma(\mathcal{A})', \mathbb{C})$-correspondence with the left $\sigma(\mathcal{A})'$-action given by the representation π. We may then form the tensor-product $(\sigma(\mathcal{A})', \mathbb{C})$-correspondence $E^\sigma \otimes \mathcal{H}$ as in Definition 3.1. Explicitly, the $\mathbb{C}$-valued inner product on $E^\sigma \otimes \mathcal{H}$ is given by

$$\langle \mu \otimes h,\, \mu' \otimes h' \rangle_{E^\sigma \otimes \mathcal{H}} = \langle \pi(\mu'^* \mu)h, h' \rangle_{\mathcal{H}} = h'^* \pi(\mu'^* \mu)h.$$

It follows that the first term on the right-hand side of (5.9) can be written as

$$\mathbf{y}^* H(\eta)\pi(\eta(I_E \otimes b)\zeta^*)H(\zeta)^*\mathbf{y}' = \langle (I_E \otimes b)\zeta^* \otimes H(\zeta)^*\mathbf{y}', \eta^* \otimes H(\eta)^*\mathbf{y} \rangle_{E^\sigma \otimes \mathcal{H}}. \tag{5.10}$$

If we replace b with $b^* b'$ (where b, b' are two elements of $\sigma(\mathcal{A})'$), use (5.10) and do some rearranging, we see that the equality (5.9) can be expressed in terms of inner products

$$\begin{aligned} \langle \pi(b')H(\zeta)^*\mathbf{y}', \pi(b)H(\eta)^*\mathbf{y} \rangle_{\mathcal{H}} &+ \langle (b' \otimes I_{\mathcal{U}})S(\zeta)^*\mathbf{y}', (b \otimes I_{\mathcal{U}})S(\eta)^*\mathbf{y} \rangle_{\mathcal{E}\otimes\mathcal{U}} \\ &= \langle (I_E \otimes b')\zeta^* \otimes H(\zeta)^*\mathbf{y}', (I_E \otimes b)\eta^* \otimes H(\eta)^*\mathbf{y} \rangle_{E^\sigma \otimes \mathcal{H}} \\ &\quad + \langle (b' \otimes I_{\mathcal{Y}})\mathbf{y}', (b \otimes I_{\mathcal{Y}})\mathbf{y} \rangle_{\mathcal{E}\otimes\mathcal{Y}}. \end{aligned} \tag{5.11}$$

Introduce subspaces $\mathcal{D}_V \subset (E^\sigma \otimes \mathcal{H}) \oplus (\mathcal{E} \otimes \mathcal{Y})$ and $\mathcal{R}_V \subset \mathcal{H} \oplus (\mathcal{E} \otimes \mathcal{U})$ by

$$\mathcal{D}_V = \overline{\operatorname{span}}\left\{ \begin{bmatrix} (I_E \otimes b)\eta^* \otimes H(\eta)^*\mathbf{y} \\ (b \otimes I_{\mathcal{Y}})\mathbf{y} \end{bmatrix} : \mathbf{y} \in \mathcal{E} \otimes \mathcal{Y}, \eta \in \mathbb{D}((E^\sigma)^*), b \in \sigma(\mathcal{A})' \right\}$$

$$\mathcal{R}_V = \overline{\operatorname{span}}\left\{ \begin{bmatrix} \pi(b)H(\zeta)^*\mathbf{y} \\ (b \otimes I_{\mathcal{U}})S(\eta)^*\mathbf{y} \end{bmatrix} : \mathbf{y} \in \mathcal{E} \otimes \mathcal{Y}, \eta \in \mathbb{D}((E^\sigma)^*), b \in \sigma(\mathcal{A})' \right\}.$$

Note that both $\mathcal{D}_V$ and $\mathcal{R}_V$ are invariant under the left action of $\sigma(\mathcal{A})'$ on $(E^\sigma \otimes \mathcal{H}) \oplus (\mathcal{E} \otimes \mathcal{Y})$ and on $\mathcal{H} \oplus (\mathcal{E} \otimes \mathcal{U})$ respectively, i.e., $\mathcal{D}_V$ and $\mathcal{R}_V$ are $\sigma(\mathcal{A})'$-submodules of $(E^\sigma \otimes \mathcal{H}) \oplus (\mathcal{E} \otimes \mathcal{Y})$ and $\mathcal{H} \oplus (\mathcal{E} \otimes \mathcal{U})$ respectively. The import of (5.11) is that the formula

$$V\colon \begin{bmatrix} (I_E \otimes b)\eta^* \otimes H(\eta)^*\mathbf{y} \\ (b \otimes I_\mathcal{Y})\mathbf{y} \end{bmatrix} \mapsto \begin{bmatrix} \pi(b)H(\zeta)^*\mathbf{y} \\ (b \otimes I_\mathcal{U})S(\eta)^*\mathbf{y} \end{bmatrix} \tag{5.12}$$

extends by linearity and continuity to a well-defined unitary operator from $\mathcal{D}_V$ onto $\mathcal{R}_V$. One easily checks that

$$V(b \cdot d) = b \cdot Vd$$

for $b \in \sigma(\mathcal{A})'$ and $d \in \mathcal{D}_V$.

By restricting in (5.12) to $b = I_\mathcal{E} \in \sigma(\mathcal{A})'$ and $\eta = 0 \in \mathbb{D}((E^\sigma)^*)$ we see that $\{0\} \oplus (\mathcal{E} \otimes \mathcal{Y}) \subset \mathcal{D}_V$. In particular

$$\mathcal{X} := ((E^\sigma \otimes \mathcal{H}) \oplus (\mathcal{E} \otimes \mathcal{Y})) \ominus \mathcal{D}_V \subset (E^\sigma \otimes \mathcal{H}) \oplus \{0\}.$$

Moreover, because $\mathcal{D}_V$ is invariant under the left $\sigma(\mathcal{A})'$-action we can see $\mathcal{X}$ as a $(\sigma(\mathcal{A})', \mathbb{C})$-correspondence, where the left action is obtained by restricting the left action on $E^\sigma \otimes \mathcal{H}$ to $\mathcal{X}$. Hence we can form the $(\sigma(\mathcal{A})', \mathbb{C})$-correspondence $\mathcal{K} = \mathcal{H} \oplus (\mathcal{F}^2(E^\sigma) \otimes \mathcal{X})$. Note that

$$E^\sigma \otimes \mathcal{K} = E^\sigma \otimes (\mathcal{H} \oplus (\mathcal{F}^2(E^\sigma) \otimes \mathcal{X})) = (E^\sigma \otimes \mathcal{H}) \oplus (E^\sigma \otimes \mathcal{F}^2(E^\sigma) \otimes \mathcal{X}).$$

So we can define an operator $\mathbf{U}$ from $\mathcal{K} \oplus (\mathcal{E} \otimes \mathcal{U})$ to $(E^\sigma \otimes \mathcal{K}) \oplus (\mathcal{E} \otimes \mathcal{Y})$ via

$$\mathbf{U}^* = \begin{bmatrix} VP_{\mathcal{D}_V} & 0 & 0 & \cdots \\ P_\mathcal{X} & 0 & 0 & \cdots \\ 0 & I_{E^\sigma \otimes \mathcal{X}} & 0 & \cdots \\ 0 & 0 & I_{(E^\sigma)^{\otimes 2} \otimes \mathcal{X}} & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} : \begin{bmatrix} (E^\sigma \otimes \mathcal{H}) \oplus (\mathcal{E} \otimes \mathcal{Y}) \\ E^\sigma \otimes \mathcal{X} \\ (E^\sigma)^{\otimes 2} \otimes \mathcal{X} \\ \vdots \\ \vdots \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \oplus (\mathcal{E} \otimes \mathcal{U}) \\ \mathcal{X} \\ E^\sigma \otimes \mathcal{X} \\ (E^\sigma)^{\otimes 2} \otimes \mathcal{X} \\ \vdots \end{bmatrix}. \tag{5.13}$$

Here $P_{\mathcal{D}_V}$ and $P_\mathcal{X}$ stand for the projections onto $\mathcal{D}_V$ and $\mathcal{X}$ respectively. One easily checks that $\mathbf{U}^*$ is an isometric $\sigma(\mathcal{A})'$-module map. In other words, $\mathbf{U}$ is a coisometry, and a $\sigma(\mathcal{A})'$-module map. The construction in (5.13) is closely related to the dilation result in [31]; see also Section 3 in [34] for more details.

Next we decompose $\mathbf{U}$ as follows:

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \otimes \mathcal{U} \end{bmatrix} \to \begin{bmatrix} E^\sigma \otimes \mathcal{H} \\ \mathcal{E} \otimes \mathcal{Y} \end{bmatrix}.$$

The definition of V and the construction of $\mathbf{U}$ imply that

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} ((I_\mathcal{E} \otimes b)\eta^*) \otimes H(\eta)^*\mathbf{y} \\ (b \otimes I_\mathcal{Y})\mathbf{y} \end{bmatrix} = \begin{bmatrix} \pi(b)H(\eta)^*\mathbf{y} \\ (b \otimes I_\mathcal{U})S(\eta)^*\mathbf{y} \end{bmatrix}.$$

By specifying this for $b = I_\mathcal{E}$ and observing that

$$\eta^* \otimes H(\eta)^*\mathbf{y} = L_{\eta^*} H(\eta)^*\mathbf{y},$$

we get

$$A^*L_{\eta^*}H(\eta)^* + C^* = H(\eta)^* \quad \text{and} \quad B^*L_{\eta^*}H(\eta)^* + D^* = S(\eta)^*. \tag{5.14}$$

Moreover, for $h \in \mathcal{H}$ we have

$$\begin{aligned} \|L_{\eta^*}h\|^2 &= \|\eta^* \otimes h\|^2 = \langle \eta^* \otimes h, \eta^* \otimes h\rangle = \langle \pi(\eta\eta^*) \otimes h, h\rangle = \|\pi(\eta\eta^*)^{\frac{1}{2}} h\|^2 \\ &\leq \|\pi(\eta\eta^*)^{\frac{1}{2}}\|^2\|h\|^2 \leq \|(\eta\eta^*)^{\frac{1}{2}}\|^2\|h\|^2 = \|\eta\|^2\|h\|^2. \end{aligned}$$

This proves that $\|L_\eta\| \leq \|\eta\| < 1$. Hence $I - A^*L_\eta$ is invertible and (5.14) shows that

$$H(\eta)^* = (I_{\mathcal{K}} - A^*L_\eta)^{-1}C^*,$$

and thus,

$$S(\eta)^* = D^* + B^*L_\eta(I_{\mathcal{K}} - A^*L_\eta)^{-1}C^*.$$

By taking adjoints we arrive at (5.5).

(3) $\Longrightarrow$ (2): Assume that (3) holds. We prove that $\mathbb{K}_S$ admits an Agler decomposition as in (5.8) with $H(\eta) := C(I - L^*_{\eta^*}A)^{-1}$. That this is equivalent to the complete positivity of the kernel $\mathbb{K}_S$ can be seen via the change of variable used in the derivation of (5.8). The fact that $\mathbf{U}$ is a coisometric $\sigma(\mathcal{A})'$-module map can also be written as

$$\begin{aligned} D(b \otimes I_{\mathcal{Y}})D^* + C\pi(b)C^* &= b \otimes I_{\mathcal{Y}}, \qquad & B(b \otimes I_{\mathcal{U}})D^* &= -A\pi(b)C^*, \\ A\pi(b)A^* + B(b \otimes I_{\mathcal{U}})B^* &= (I_E \otimes b) \otimes I_{\mathcal{K}}, \qquad & D(b \otimes I_{\mathcal{U}})B^* &= -C\pi(b)A^*. \end{aligned}$$

Note that

$$H(\eta) = C(I - L^*_{\eta^*}A)^{-1} = C + C(I - L^*_{\eta^*}A)^{-1}L^*_{\eta^*}A = C + H(\eta)L^*_{\eta^*}A,$$

and

$$S(\eta) = D + H(\eta)L^*_{\eta^*}B.$$

Hence

$$\begin{aligned} H(\eta)\pi(b)H(\zeta)^* &= (C + H(\eta)L^*_{\eta^*}A)\pi(b)(C^* + A^*L_{\zeta^*}H(\zeta)^*) \\ &= C\pi(b)C^* + C\pi(b)A^*L_{\zeta^*}H(\zeta)^* + H(\eta)L^*_{\eta^*}A\pi(b)C^* \\ &\quad + H(\eta)L^*_{\eta^*}A\pi(b)A^*L_{\zeta^*}H(\zeta)^* \\ &= b \otimes I_{\mathcal{Y}} - D(b \otimes I_{\mathcal{U}})D^* - D(b \otimes I_{\mathcal{U}})B^*L_{\zeta^*}H(\zeta)^* \\ &\quad - H(\eta)L^*_{\eta^*}B(b \otimes I_{\mathcal{U}})D^* - H(\eta)L^*_{\eta^*}B(b \otimes I_{\mathcal{U}})B^*L_{\zeta^*}H(\zeta)^* \\ &\quad + H(\eta)L^*_{\eta^*}((I_E \otimes b) \otimes I_{\mathcal{H}})L_{\zeta^*}H(\zeta)^* \\ &= b \otimes I_{\mathcal{Y}} - D(b \otimes I_{\mathcal{U}})S(\zeta)^* - H(\eta)L^*_{\eta^*}B(b \otimes I_{\mathcal{U}})S(\zeta)^* \\ &\quad + H(\eta)\pi(\eta(I_E \otimes b)\zeta^*)H(\zeta)^* \\ &= b \otimes I_{\mathcal{Y}} - S(\eta)(b \otimes I_{\mathcal{U}})S(\zeta)^* + H(\eta)\pi(\eta(I_E \otimes b)\zeta^*)H(\zeta)^*. \end{aligned}$$

In this way we have proved that (5.8) holds.

(2) $\Longrightarrow$ (1′): Assume that (2) holds. Consider the formula

$$(M_S)^* \colon b^* \cdot (k_{E,\sigma;\zeta} \otimes I_{\mathcal{Y}})\mathbf{y} \mapsto b^* \cdot (k_{E,\sigma;\zeta} \otimes I_{\mathcal{U}})S(\zeta)^*\mathbf{y} \tag{5.15}$$

for $b \in \sigma(\mathcal{A})'$, $\zeta \in \mathbb{D}((E^\sigma)^*)$ and $\mathbf{y} \in \mathcal{E} \otimes \mathcal{Y}$. Then the complete positivity of the kernel $\mathbb{K}_S$ is exactly what is needed to see that the formula (5.15) can be extended by linearity and continuity to define a contraction operator $(M_S)^*$ from $H^2_{\mathcal{Y}}(E,\sigma)$ into $H^2_{\mathcal{U}}(E,\sigma)$ which is also a $\sigma(\mathcal{A})'$-module map:

$$b^* \cdot (M_S^* f) = M_S^*(b^* \cdot f) \text{ for all } b \in \sigma(\mathcal{A})' \quad \text{and} \quad f \in H^2_{\mathcal{Y}}(E,\sigma).$$

Here we are using that the span of the collection of kernel functions

$$\{b^* \cdot (k_{E,\sigma;\zeta} \otimes I_{\mathcal{Y}})\mathbf{y} \colon b \in \sigma(\mathcal{A})',\, \zeta \in \mathbb{D}((E^\sigma)^*), \mathbf{y} \in \mathcal{E} \otimes \mathcal{Y}\}$$

is dense in $H^2_{\mathcal{Y}}(E,\sigma)$. Then the computation

$$\begin{aligned}
\langle (M_S f)(\zeta,b), \mathbf{y}\rangle_{\mathcal{E}\otimes\mathcal{Y}} &= \langle M_S f, b^* \cdot (k_{E,\sigma;\zeta} \otimes I_{\mathcal{Y}})\mathbf{y}\rangle_{H^2_{\mathcal{Y}}(E,\sigma)}\\
&= \langle f, M_S^*(k_{E,\sigma;\zeta} \otimes I_{\mathcal{Y}})\mathbf{y}\rangle_{H^2_{\mathcal{U}}(E,\sigma)}\\
&= \langle f, b^* \cdot (k_{E,\sigma;\zeta} \otimes I_{\mathcal{Y}})S(\zeta)^*\mathbf{y}\rangle_{H^2_{\mathcal{U}}(E,\sigma)}\\
&= \langle f(\zeta,b), S(\zeta)^*\mathbf{y}\rangle_{\mathcal{E}\otimes\mathcal{U}}\\
&= \langle S(\zeta) f(\zeta,b), \mathbf{y}\rangle_{\mathcal{E}\otimes\mathcal{Y}}
\end{aligned}$$

shows that M_S is indeed the operator of multiplication by S. □

6. Examples

In this section we illustrate the general theory for some more concrete special cases. For simplicity we consider here only examples of the theory developed in Sections 3, 4 and 5 with $\mathcal{U} = \mathcal{Y} = \mathbb{C}$. Unlike what one might expect, this does not lead to scalar versions of the results discussed in Sections 1 and 2, but rather to square versions, that is, we regain Theorems 1.1, 2.1 and 2.3 for the case $\mathcal{U} = \mathcal{Y}$, but not necessarily equal to $\mathbb{C}$.

6.1. The classical case

In this example, we take $\mathcal{A} = \mathcal{L}(\mathcal{G})$ for a given Hilbert space $\mathcal{G}$. Let $E = \mathcal{L}(\mathcal{G})$ viewed a correspondence over itself in the standard way:

$$a \cdot \xi = a\xi, \quad \xi \cdot a = \xi a \text{ (the operator multiplication in } \mathcal{L}(\mathcal{G})) \text{ for } a \in \mathcal{A},\, \xi \in E,$$
$$\langle \xi', \xi\rangle = \xi^* \xi' \text{ for } \xi', \xi \in E.$$

Note that the inner product is the $\mathcal{L}(\mathcal{G})$-inner product when considered as a correspondence over itself. Since

$$\xi_n \otimes \xi_{n-1} \otimes \cdots \otimes \xi_1 = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \xi_n \cdots \xi_1,$$

we can identify $E^{\otimes n}$ with $E = \mathcal{L}(\mathcal{G})$ and then the Fock space $\mathcal{F}^2(E)$ has the form

$$\mathcal{F}^2(E) = \oplus_{n=0}^\infty E^{\otimes n} = \mathcal{L}(\mathcal{G}, \ell^2_{\mathcal{G}}(\mathbb{Z}_+)).$$

The abstract analytic Toeplitz algebra $\mathcal{F}^\infty(E)$ is the collection of all lower triangular Toeplitz matrices with $\mathcal{L}(\mathcal{G})$-block entries acting as bounded operators on $\mathcal{L}(\mathcal{G}, \ell^2_{\mathcal{G}}(\mathbb{Z}_+))$ (and thus bounded on $\ell_{\mathcal{G}}(\mathbb{Z}_+)$).

Now suppose we are given a Hilbert space $\mathcal{E}_0$, let $\mathcal{E} = \mathcal{G} \otimes \mathcal{E}_0$ and σ be the representation of $\mathcal{A} = \mathcal{L}(\mathcal{G})$ on $\mathcal{L}(\mathcal{E})$ given by $\sigma(a) = a \otimes I_{\mathcal{E}_0}$.Then

$$\begin{aligned} \sigma(\mathcal{A})' &= \{b \in \mathcal{L}(\mathcal{E})\colon b\sigma(a) = \sigma(a)b \text{ for all } a \in \mathcal{A}\} \\ &= \{b \in \mathcal{L}(\mathcal{E})\colon b(a^0 \otimes I_{\mathcal{E}_0}) = (a^0 \otimes I_{\mathcal{E}_0})b \text{ for all } a^0 \in \mathcal{L}(\mathcal{G})\} \\ &= \{I_{\mathcal{G}} \otimes b^0 : b^0 \in \mathcal{L}(\mathcal{E}_0)\} \end{aligned}$$

and hence $\sigma(\mathcal{A})'$ can be identified with $\mathcal{L}(\mathcal{E}_0)$.

We next note that

$$\mathcal{F}^2(E,\sigma) = \mathcal{F}^2(E) \otimes_\sigma \mathcal{E} = \mathcal{L}(\mathcal{G}, \ell^2_{\mathcal{G}}(\mathbb{Z}_+)) \otimes_\sigma (\mathcal{G} \otimes \mathcal{E}_0) = \ell^2_{\mathcal{G}}(\mathbb{Z}_+) \otimes \mathcal{E}_0 = \ell^2_{\mathcal{E}}(\mathbb{Z}_+).$$

The representations $\varphi_{\infty,\sigma}\colon \mathcal{A} = \mathcal{L}(\mathcal{G}) \to \mathcal{L}(\ell^2_{\mathcal{E}}(\mathbb{Z}_+))$ and $\iota_{\infty,\sigma}\colon \sigma(\mathcal{A})' = \mathcal{L}(\mathcal{E}_0) \to \mathcal{L}(\ell^2_{\mathcal{E}}(\mathbb{Z}_+))$ are given by

$$\varphi_{\infty,\sigma}(a) = I_{\ell^2(\mathbb{Z}^+)} \otimes a \otimes I_{\mathcal{E}_0}, \qquad \iota_{\infty,\sigma}(b^0) = I_{\ell^2(\mathbb{Z}^+)} \otimes I_{\mathcal{G}} \otimes b^0.$$

We next compute

$$\begin{aligned} (E^\sigma)^* &= \{\eta\colon E \otimes_\sigma \mathcal{E} \to \mathcal{E}\colon \eta(a \otimes I_{\mathcal{E}}) = \sigma(a)\eta, a \in \mathcal{L}(\mathcal{G})\} \\ &= \{\eta\colon \mathcal{L}(\mathcal{G}) \otimes_\sigma \mathcal{G} \otimes \mathcal{E}_0 \to \mathcal{G} \otimes \mathcal{E}_0\colon \eta(a \otimes I_{\mathcal{E}}) = (a \otimes I_{\mathcal{E}_0})\eta, a \in \mathcal{L}(\mathcal{G})\} \\ &= \{\eta\colon \mathcal{G} \otimes \mathcal{E}_0 \to \mathcal{G} \otimes \mathcal{E}_0\colon \eta(a \otimes I_{\mathcal{E}_0}) = (a \otimes I_{\mathcal{E}_0})\eta, a \in \mathcal{L}(\mathcal{G})\} \\ &= \{I_{\mathcal{G}} \otimes \eta^0 : \eta^0 \in \mathcal{L}(\mathcal{E}_0)\}. \end{aligned}$$

We conclude that $(E^\sigma)^*$ can be identified with $\mathcal{L}(\mathcal{E}_0)$.

The creation operators and dual creation operators then have the form

$$\begin{aligned} T_{\xi,\sigma} &= \mathbf{S} \otimes \xi \otimes I_{\mathcal{E}_0} \text{ for } \xi \in \mathcal{A} = \mathcal{L}(\mathcal{G}), \\ T^d_{\mu^0,\sigma} &= \mathbf{S} \otimes I_{\mathcal{G}} \otimes \mu^0 \text{ for } \mu^0 \in \mathcal{L}(\mathcal{E}^0) \cong E^\sigma \end{aligned}$$

where $\mathbf{S}$ is the standard shift operator on $\ell^2(\mathbb{Z}_+)$:

$$\mathbf{S}\colon \{c_n\}_{n\in\mathbb{Z}_+} \mapsto \{c'_n\}_{n\in\mathbb{Z}_+} \quad \text{where} \quad c'_0 = 0,\, c'_n = c_{n-1} \text{ for } n \geq 1.$$

Note that the commutativity properties laid out in Proposition 4.2 are now transparent for this example.

Then, for $f = \oplus_{n=0}^\infty f_n \in \mathcal{F}^2(E,\sigma)$, the Fourier transform $\Phi f = f^\wedge$ is given by

$$f^\wedge(\eta^0, b^0) = \sum_{n=0}^\infty (I_{\mathcal{G}} \otimes (\eta^0)^n b^0) f_n \in \mathcal{E}$$

for $\eta \in \mathbb{B}(\mathcal{L}(\mathcal{E}_0))$ (the open unit ball of $\mathcal{L}(\mathcal{E}_0)$) and $b^0 \in \mathcal{L}(\mathcal{E}_0)$. One can check that Φ is injective. It follows that Φ is a unitary transformation from $\ell^2_{\mathcal{E}}(\mathbb{Z}_+)$ onto a Hilbert space $H^2(E,\sigma)$ of $\mathcal{E}$-valued functions on $\mathbb{B}(\mathcal{L}(\mathcal{E}_0)) \times \mathcal{L}(\mathcal{E}_0)$ carrying a $\mathcal{L}(\mathcal{E}_0)$-representation:

$$\pi_{H^2(E,\sigma)}(b^0)\colon f^\wedge(\eta'^0, b'^0) \mapsto f^\wedge(\eta'^0, b'^0 b^0).$$

In fact $f^\wedge(\eta^0, I_{\mathcal{E}_0}) = 0$ for all $\eta^0 \in \mathcal{B}(\mathcal{L}(\mathcal{E}_0))$ already forces f to be zero in $\ell^2_{\mathcal{E}}(\mathbb{Z}_+)$ so the function $f^\wedge$ is determined completely by its single-variable restriction $f^{\wedge 1} :=$

$f^\wedge(\cdot, I_{\mathcal{E}_0})$ and one can work with the space $\widetilde{H}^2(E,\sigma) = \{f^{\wedge 1}\colon f \in \ell^2_{\mathcal{E}}\}$ instead. One can identify $\widetilde{H}^2(E,\sigma)$ with functions of the form

$$g(\eta^0) = \sum_{n=0}^{\infty} (I_{\mathcal{G}} \otimes (\eta^0)^n) g_n$$

where $\oplus_{n=0}^{\infty} g_n \in \ell^2_{\mathcal{E}}(\mathbb{Z}_+)$ with $\|g\|_{\widetilde{H}^2(E,\sigma)} = \|\oplus_{n=0}^{\infty} g_n\|_{\ell^2_{\mathcal{E}}(\mathbb{Z}_+)}$ and with the $\sigma(\mathcal{A})' \cong \mathcal{L}(\mathcal{E}_0)$-left action given by

$$(b^0 \cdot g)(\eta) = \sum_{n=0}^{\infty} (I_{\mathcal{G}} \otimes (\eta^0)^n)(I \otimes b^0) g_n \text{ if } g(\eta^0) = \sum_{n=0}^{\infty} (I_{\mathcal{G}} \otimes (\eta^0)^n) g_n.$$

An element S of $\mathcal{F}^\infty(E,\sigma)$ is an operator on $\ell^2_{\mathcal{E}}(\mathbb{Z}_+)$ having a lower-triangular Toeplitz matrix representation

$$R = [R_{i-j}]_{i,j=0,1,\dots}$$

where each R_n is an operator on $\mathcal{E}$ of the form $R_n = R_n^0 \otimes I_{\mathcal{E}_0}$ for an operator $R_n^0 \in \mathcal{L}(\mathcal{G})$ with $R_n^0 = 0$ for $n < 0$. Given such an R, the associated $\mathcal{L}(\mathcal{E})$-valued function $R^\wedge\colon \mathcal{L}(\mathcal{E}_0) \to \mathcal{L}(\mathcal{E})$ is then given by

$$R^\wedge(\eta^0) = \sum_{n=0}^{\infty} R_n^0 \otimes (\eta^0)^n.$$

The Schur class $\mathcal{S}(E,\sigma)$ for this case can be identified with the set of functions $S\colon \mathbb{B}(\mathcal{L}(\mathcal{E}_0)) \to \mathcal{L}(\mathcal{E})$ with a presentation of the form

$$S(\eta^0) = \sum_{n=0}^{\infty} S_n^0 \otimes (\eta^0)^n \tag{6.1}$$

for which the associated Toeplitz matrix

$$[S^0_{i-j}]_{i,j=0,1,\dots}$$

defines a contraction operator on $\ell^2_{\mathcal{G}}(\mathbb{Z}_+)$. If we use the single-variable version $\widetilde{H}^2(E,\sigma)$ of the Hardy space, the condition in part (2) of Theorem 5.1 means not only that

$$M_S\colon f^{\wedge 1}(\eta^0) \mapsto S(\eta^0) f^{\wedge 1}(\eta^0)$$

maps $\widetilde{H}^2(E,\sigma)$ contractively into $\widetilde{H}^2(E,\sigma)$, but also that M_S is a $\mathcal{L}(\mathcal{E}_0)$-module map:

$$M_S(b \cdot f^{\wedge 1}) = b \cdot M_S f^{\wedge 1}.$$

The realization formula (5.4) and (5.5) from part (3) of Theorem 5.1 tells us that such functions S are characterized by having a realization of the form

$$S(\eta^0) = D + C(I - \pi(\eta^0)A)^{-1}\pi(\eta^0)B \tag{6.2}$$

where

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix}$$

is a unitary operator and π is a $*$-representation of $\mathcal{L}(\mathcal{E}_0)$ to $\mathcal{L}(\mathcal{H})$ which is also a $\mathcal{L}(\mathcal{E}_0)$-module map:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \pi(b^0) & 0 \\ 0 & I_{\mathcal{G}} \otimes b^0 \end{bmatrix} = \begin{bmatrix} \pi(b^0) & 0 \\ 0 & I_{\mathcal{G}} \otimes b^0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{6.3}$$

Here we use that $E^\sigma \otimes_\pi \mathcal{H}$ can be identified with $\mathcal{H}$ since $(I_{\mathcal{G}} \otimes (\eta^0)^*) \otimes h = I_{\mathcal{E}} \otimes \pi((\eta^0)^*)h$.

We note that is easy to see that a realization as in (6.2) implies that S has a presentation of the form (6.1). Indeed, if U is unitary and satisfies (6.3), since A commutes with $\pi(\eta^0)$ we see that $(\pi(\eta^0)A)^n = A^n \pi(\eta^0)^n$. Hence expansion of the inverse in (6.2) as a geometric series and repeated usage of (6.3) gives

$$S(\eta^0) = \sum_{n=0}^{\infty} S_n (I \otimes (\eta^0)^n)$$

where

$$S_0 = D, \qquad S_n = CA^{n-1}B \text{ for } n \geq 1.$$

Additional usage of (6.3) gives us

$$S_n(I \otimes \eta^0) = (I \otimes \eta^0) S_n \text{ for all } \eta^0 \in \mathcal{L}(\mathcal{E}_0)$$

from which we conclude that S_n has the form $S_n = S_n^0 \otimes I_{\mathcal{E}_0}$ for operators S_n^0 acting on $\mathcal{G}$, and hence $S(\eta^0)$ has the form as in (6.1).

Conversely, if $S \colon \mathbb{B}(\mathcal{L}(\mathcal{E}_0)) \to \mathcal{L}(\mathcal{E})$ is of the form (6.1), it follows that $S^0(\lambda) = \sum_{n=0}^{\infty} S_n^0 \lambda^n$ is in the classical Schur class $\mathcal{S}(\mathcal{G}, \mathcal{G})$. By the classical realization theorem we can write

$$S^0(\lambda) = D^0 + \lambda C^0 (I - \lambda A^0)^{-1} B^0$$

where

$$\mathbf{U}^0 = \begin{bmatrix} A^0 & B^0 \\ C^0 & D^0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}^0 \\ \mathcal{G} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}^0 \\ \mathcal{G} \end{bmatrix}$$

is coisometric (or even unitary). Then

$$\mathbf{U} = \mathbf{U}^0 \otimes I_{\mathcal{E}^0} = \begin{bmatrix} A^0 \otimes I_{\mathcal{E}_0} & B \otimes I_{\mathcal{E}_0} \\ C^0 \otimes I_{\mathcal{E}_0} & D^0 \otimes I_{\mathcal{E}_0} \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix}$$

(where we set $\mathcal{H} = \mathcal{H}^0 \otimes \mathcal{E}_0$) with

$$\pi(b) = I_{\mathcal{H}^0} \otimes b \in \mathcal{L}(\mathcal{H}) \text{ for } b \in \mathcal{L}(\mathcal{E}^0)$$

provides a realization for S as in (6.2). Thus the general theory provides a new kind of realization result, but one can easily derive this result directly from the classical realization theorem.

Two special cases of the above analysis are of interest.

1. If we take $\mathcal{G} = \mathcal{E}, \mathcal{E}_0 = \mathbb{C}$ in the example, we have $\mathcal{F}^2(E) = \mathcal{L}(\mathcal{E}, \ell^2_{\mathcal{E}}(\mathbb{Z}_+))$ with $\mathcal{F}^\infty(E)$ equal to the collection of all lower triangular Toeplitz matrices with $\mathcal{L}(\mathcal{E})$-block entries acting on $\ell^2_{\mathcal{E}}(\mathbb{Z}_+)$. In this case $\sigma(\mathcal{A})' = \mathbb{C}I_{\mathcal{E}}$. and $(E^\sigma)^* = I_{\mathcal{E}} \otimes \mathbb{C}$ is isomorphic to $\mathbb{C}$; thus $\mathbb{D}((E^\sigma)^*)$ may be identified with the

open unit disk $\mathbb{D}$ of $\mathbb{C}$. Moreover $\mathcal{F}(E) \otimes_\sigma \mathcal{E} = \ell^2_{\mathcal{E}}(\mathbb{Z}_+)$ and for a given $\lambda \in \mathbb{D}$, we have the bounded point-evaluation:

$$f = \oplus_{n=0}^\infty f_n \in \mathcal{F}(E) \otimes_\sigma \mathcal{E} = \ell^2_{\mathcal{E}}(\mathbb{Z}_+) \to \widehat{f}(\lambda) = \sum_{n=0}^\infty f_n \lambda^n \in H^2_{\mathcal{E}}(\mathbb{D}).$$

Then

$$H^2(E, \sigma) = H^2_{\mathcal{E}}(\mathbb{D})$$

and

$$H^\infty(E, \sigma) = H^\infty_{\mathcal{L}(\mathcal{E})}(\mathbb{D}) \otimes I_{\mathcal{E}} = H^\infty_{\mathcal{L}(\mathcal{E})}(\mathbb{D}).$$

Hence $S \in H^\infty(E, \sigma)$ means, for $\lambda \in \mathbb{D}$, that $S(\lambda) = \sum_{n=0}^\infty S_n \lambda^n$ with $S_n \in \mathcal{L}(\mathcal{E})$. The operators in $H^\infty(E, \sigma)$ with norm at most equal to 1 form the classical Schur class. If we apply the general theorem 5.1 for this case, we simply recover Theorem 1.1 (where $\mathcal{U} = \mathcal{Y} = \mathcal{G}$).

2. If we take $\mathcal{G} = \mathbb{C}, \mathcal{E}_0 = \mathcal{E}$, then $\mathcal{F}^2(E) = \ell^2(\mathbb{Z}_+)$, $\mathcal{F}^\infty(E)$ is the collection of all lower triangular Toeplitz matrices acting on $\ell^2(\mathbb{Z}_+)$. In this case $\sigma(\mathcal{A})' = \mathcal{L}(\mathcal{E})$ and

$$(E^\sigma)^* = \{\eta \colon \mathbb{C} \otimes_\sigma \mathcal{E} \to \mathcal{E} \colon \eta(a \otimes I_{\mathcal{E}}) = a\eta, a \in \mathbb{C}\}.$$

Since $\mathbb{C} \otimes_\sigma \mathcal{E}$ can be identified with $\mathcal{E}$ in the obvious way, $(E^\sigma)^*$ amounts to $\mathcal{L}(\mathcal{E})$. We also have $\mathcal{F}^2(E) \otimes_\sigma \mathcal{E} = \ell^2_{\mathcal{E}}(\mathbb{Z}_+)$. For a given $\eta \in \mathbb{D}((E^\sigma)^*) = \mathbb{B}(\mathcal{L}(\mathcal{E}))$ and $b \in \sigma(\mathcal{A})' = \mathcal{L}(\mathcal{E})$ we have the bounded point evaluation:

$$f = \oplus_{n=0}^\infty f_n \in \mathcal{F}(E) \otimes_\sigma \mathcal{E} = \ell^2_{\mathcal{E}}(\mathbb{Z}_+) \mapsto f^\wedge(\eta, b) = \sum_{n=0}^\infty \eta^n b f_n.$$

We may view this $H^2(E, \sigma)$ simply as functions of the form $\eta \mapsto \sum_{n=0}^\infty \eta^n f_n$ with $\oplus_{n=0}^\infty f_n \in \ell^2_{\mathcal{E}}$ which also carries an $\mathcal{L}(\mathcal{E})$-action:

$$f(\eta) \mapsto (b \cdot f)(\eta) = \sum_{n=0}^\infty \eta^n b f_n \text{ if } f(\eta) = \sum_{n=0}^\infty \eta^n f_n$$

or

$$b \cdot f = \sum_{n=0}^\infty \mathbf{S}^n b P_{\mathcal{E}} \mathbf{S}^{*n} f$$

where $P_{\mathcal{E}}$ is the projection onto the constant functions and $\mathbf{S}$ is the operator-argument shift operator $(\mathbf{S}f)(\eta) = \eta \cdot f(\eta)$. If we identify $S = S^0 \otimes I_{\mathcal{E}} \in H^\infty(E, \sigma) = H^\infty \otimes I_{\mathcal{E}}$ with the scalar-valued function $S^0 \in H^\infty(\mathbb{D})$, then the associated function with operator argument

$$\eta \mapsto S^0(\eta) = \sum_{n=0}^\infty \eta^n (S^0_n \otimes I_{\mathcal{E}}) \in \mathcal{L}(\mathcal{E}) \tag{6.4}$$

corresponds to the functional calculus for scalar holomorphic functions with operator argument usually defined via the holomorphic functional calculus (see, e.g., [48]). The positivity of the kernel

$$K_S(\eta, \zeta) = (I - \eta\zeta^*)^{-1} - S(\zeta)(I - \eta\zeta^*)^{-1} S(\eta)^*$$

guarantees that the multiplication operator

$$M_S\colon f(\eta) \mapsto S(\eta)f(\eta)$$

is contractive on $H^2(E,\sigma)$ while complete positivity of the enlarged kernel

$$\mathbb{K}_S(\eta,\zeta)[b] = \sum_{n=0}^{\infty} \eta^n b \zeta^{*n} - S(\eta)\left(\sum_{n=0}^{\infty} \eta^n b \zeta^{*n}\right) S(\zeta)^*$$

guarantees in addition that S has the form (6.4) and that the associated multiplication operator M_S commutes with the $\mathcal{L}(\mathcal{E})$-action:

$$b \cdot (M_S f) = M_S(b \cdot f) \text{ for } b \in \mathcal{L}(\mathcal{E}),\ f \in H^2(e,\sigma).$$

The realization result (the equivalence of (1) and (3) in Theorem 5.1) follows from the classical realization result for scalar-valued Schur-class functions in the same way as was explained above for the general case of this example.

6.2. Free semigroup algebras

In this example, we take $\mathcal{A} = \mathcal{L}(\mathcal{G})$ for a given Hilbert space $\mathcal{G}$ and E to be the d-fold column space $\operatorname{col}_{j=1}^{d} \mathcal{L}(\mathcal{G})$ over $\mathcal{L}(\mathcal{G})$ viewed as a correspondence over $\mathcal{L}(\mathcal{G})$ in the standard way (see [31, 43]):

$$a \cdot \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix} = \begin{bmatrix} a\xi_1 \\ \vdots \\ a\xi_d \end{bmatrix}, \qquad \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix} \cdot a = \begin{bmatrix} \xi_1 a \\ \vdots \\ \xi_d a \end{bmatrix}, \qquad \left\langle \begin{bmatrix} \xi'_1 \\ \vdots \\ \xi'_d \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix} \right\rangle = \sum_{j=1}^{d} \xi_j^* \xi'_j$$

$$\text{for } \xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix},\ \xi' = \begin{bmatrix} \xi'_1 \\ \vdots \\ \xi'_d \end{bmatrix} \in E \quad \text{and} \quad a \in \mathcal{L}(\mathcal{G}). \tag{6.5}$$

One can then identify $E^{\otimes n}$ with the column space $\mathcal{L}(\mathcal{G}, \oplus_{\alpha\colon |\alpha|=n}\mathcal{G})$ where $\alpha = i_n \cdots i_1$ is in the free semigroup $\mathcal{F}_d$ with notation as in Subsection 2.2. Then the associated Fock space is

$$\mathcal{F}^2(E) = \oplus_{n=0}^{\infty} E^{\otimes n} = \mathcal{L}(\mathcal{G}, \oplus_{n=0}^{\infty} \oplus_{\alpha\colon |\alpha|=n} \mathcal{G})$$

can equally well be viewed as

$$\mathcal{F}^2(E) = \mathcal{L}(\mathcal{G}, \ell^2_{\mathcal{G}}(\mathcal{F}_d)).$$

Then the analytic Toeplitz algebra $\mathcal{F}^\infty(E)$ can be identified as

$$\mathcal{F}^\infty(E) = \mathcal{L}_d \otimes \mathcal{L}(\mathcal{G}),$$

where $\mathcal{L}_d$ is the free semigroup algebra discussed by Davidson and Pitts in [21] and is also the ultraweak closure of Popescu's noncommutative disk algebra (see [38]).

Just as done in the discussion of the classical case above, we now let $\mathcal{E}_0$ be another Hilbert space and take $\mathcal{E}$ to be $\mathcal{E} = \mathcal{G} \otimes \mathcal{E}_0$. We consider the $*$-representation σ of $\mathcal{A} = \mathcal{L}(\mathcal{G})$ to $\mathcal{L}(\mathcal{E})$ given by

$$\sigma(a) = a \otimes I_{\mathcal{E}_0} \text{ for } a \in \mathcal{L}(\mathcal{G}).$$

We compute $\sigma(\mathcal{A})'$ as follows:

$$\begin{aligned}\sigma(\mathcal{A})' &= \{b \in \mathcal{L}(\mathcal{E})\colon b\sigma(a) = \sigma(a)e \text{ for all } a \in \mathcal{A}\}\\ &= \{b \in \mathcal{L}(\mathcal{E})\colon b(a \otimes I_{\mathcal{E}_0}) = (a \otimes I_{\mathcal{E}_0})b \text{ for all } a \in \mathcal{L}(\mathcal{G})\}\\ &= \{I_{\mathcal{G}} \otimes b^0 : b^0 \in \mathcal{L}(\mathcal{E}_0)\}.\end{aligned}$$

Hence $\sigma(\mathcal{A})'$ can be identified with $\mathcal{L}(\mathcal{E}_0)$.

We next note that

$$E^{\otimes n} \otimes_\sigma \mathcal{E} = \operatorname{col}_{\alpha\colon |\alpha|=n} \mathcal{L}(\mathcal{G}) \otimes_\sigma \mathcal{E} \cong \operatorname{col}_{\alpha\colon |\alpha|=n} \mathcal{E}$$

and hence we identify $\mathcal{F}^2(E,\sigma) = \mathcal{F}^2(E) \otimes_\sigma \mathcal{E}$ as

$$\mathcal{F}^2(E,\sigma) = \oplus_{\alpha\in\mathcal{F}_d}\mathcal{E} = \ell^2_{\mathcal{E}}(\mathcal{F}_d).$$

The representations $\varphi_{\infty,\sigma}\colon \mathcal{L}(\mathcal{G}) = \mathcal{A} \to \mathcal{L}(\ell^2_{\mathcal{E}}(\mathcal{F}_d))$ and $\iota_{\infty,\sigma}\colon \mathcal{L}(\mathcal{E}_0) \cong \sigma(\mathcal{A})' \to \mathcal{L}(\ell^2_{\mathcal{E}}(\mathcal{F}_d))$ can be seen to be given by

$$\begin{aligned}\varphi_{\infty,\sigma}(a) &= I_{\ell^2(\mathcal{F}_d)} \otimes a \otimes I_{\mathcal{E}_0} \text{ for } a \in \mathcal{L}(\mathcal{G}) = \mathcal{A},\\ \iota_{\infty,\sigma}(b^0) &= I_{\ell^2(\mathcal{F}_d)} \otimes I_{\mathcal{G}} \otimes b^0 \text{ for } b^0 \in \mathcal{L}(\mathcal{E}_0) \cong \sigma(\mathcal{A})'.\end{aligned}$$

We now observe that $E \otimes_\sigma \mathcal{E}$ can be identified with $\mathcal{E}^d$ (the d-fold direct sum of $\mathcal{E}$ with itself) under the identification

$$\begin{bmatrix}\xi_1\\ \vdots\\ \xi_d\end{bmatrix} \otimes e \cong \begin{bmatrix}(\xi_1 \otimes I_{\mathcal{E}_0})e\\ \vdots\\ (\xi_d \otimes I_{\mathcal{E}_0})e\end{bmatrix} \text{ for } \xi_1,\dots,\xi_d \in \mathcal{L}(\mathcal{G}) \quad\text{and}\quad e \in \mathcal{E}.$$

Then $\eta \in (E^\sigma)^*$ means that η is a block row-matrix $\eta = \begin{bmatrix}\eta_1 & \cdots & \eta_d\end{bmatrix}$ mapping $E \otimes_\sigma \mathcal{E} \cong \mathcal{E}^d$ into $\mathcal{E}$ with the additional property that

$$\begin{bmatrix}\eta_1 & \cdots & \eta_d\end{bmatrix} \operatorname{diag}(a \otimes I_{\mathcal{E}_0}) = (a \otimes I_{\mathcal{E}_0}) \begin{bmatrix}\eta_1 & \cdots & \eta_d\end{bmatrix} \text{ for all } a \in \mathcal{L}(\mathcal{G}).$$

It follows that $\eta_j(a \otimes I_{\mathcal{E}_0}) = (a \otimes I_{\mathcal{E}_0})\eta_j$ and hence that $\eta_j = I_{\mathcal{G}} \otimes \eta_j^0$ for some $\eta_j^0 \in \mathcal{L}(\mathcal{E}_0)$ for $j = 1,\dots,d$ and we have an identification

$$(E^\sigma)^* \cong \mathcal{L}(\mathcal{E}_0^d, \mathcal{E}_0).$$

One can check that the creation and dual creation operators are given by

$$\begin{aligned}T_{\xi,\sigma} &= \sum_{j=1}^d \mathbf{S}_j \otimes \xi_j \otimes I_{\mathcal{E}_0} \text{ for } \xi = \begin{bmatrix}\xi_1\\ \vdots\\ \xi_d\end{bmatrix} \in \mathcal{L}(\mathcal{G})^d = E,\\ T_{\mu^0,\sigma} &= \sum_{j=1}^d \mathbf{S}_j \otimes I_{\mathcal{G}} \otimes \mu_j^0 \text{ for } \mu^0 = \begin{bmatrix}\mu_1^0\\ \vdots\\ \mu_d^0\end{bmatrix} \in \mathcal{L}(\mathcal{E}_0, \mathcal{E}_0^d) \cong E^\sigma.\end{aligned}$$

For $\eta^0 = \begin{bmatrix} \eta_1^0 & \cdots & \eta_d^0 \end{bmatrix} \in \mathbb{B}(\mathcal{L}(\mathcal{E}_0^d, \mathcal{E}_0)) \cong \mathbb{D}((E^\sigma)^*)$ and $b^0 \in \mathcal{L}(\mathcal{E}_0) \cong \sigma(\mathcal{A})'$, we have the bounded point-evaluation operator on $\mathcal{F}^2(E, \sigma) = \ell^2_{\mathcal{E}}(\mathcal{F}_d)$:

$$f = \oplus_{\alpha \in \mathcal{F}_d} f_\alpha \mapsto f^\wedge(\eta^0, b^0) := \sum_{\alpha \in \mathbb{F}_d} (I_{\mathcal{G}} \otimes (\eta^0)^\alpha b^0) f_\alpha \tag{6.6}$$

where $(\eta^0)^\alpha = \eta^0_{i_N} \cdots \eta^0_{i_1}$ for $\alpha = i_N \cdots i_1 \in \mathcal{F}_d$. To continue a detailed analysis, we now consider in turn two divergent special cases.

Case 1. $\mathcal{E}_0 = \mathbb{C}$ *so* $\mathcal{E} = \mathcal{G}$: In this case we identify $\mathbb{D}((E^\sigma)^*) \cong \mathbb{B}(\mathcal{L}(\mathcal{E}_0^d, \mathcal{E}_0))$ with the unit ball in $\mathbb{C}^d$

$$\mathbb{B}^d := \left\{ \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d \colon \sum_{j=1}^d |\lambda_j|^2 < 1 \right\}.$$

The point-evaluation map

$$f^\wedge(\lambda, b) = \sum_{n \in \mathbb{Z}^d_+} \left[\sum_{\alpha \in \mathcal{F}_d \colon |\alpha| = n} b f_\alpha \right] \lambda^n = b \cdot f^\wedge(\lambda, I_{\mathcal{E}}) \tag{6.7}$$

where we use the standard $(\mathcal{L}(\mathcal{E}), \mathbb{C})$-correspondence structure on $\mathcal{E}$. Here also we use the standard commutative multivariable notation

$$\lambda^n = \lambda_1^{n_1} \cdots \lambda_d^{n_d} \text{ if } n = (n_1, \dots, n_d) \in \mathbb{Z}^d_+.$$

From (6.7) we see that we are in the situation of Remark 3.7 and completely positivity of the kernel

$$\mathbb{K}(\lambda, \lambda')[b^* b'] = \sum_{n \in \mathbb{Z}^d_+} (\lambda^n I_{\mathcal{E}})(b^* b')(\overline{\lambda}')^n I_{\mathcal{E}})$$

associated with $H^2(E, \sigma)$ for this case reduces to classical Aronszajn positivity for the Drury-Arveson kernel

$$k(\lambda, \lambda') = \sum_{n \in \mathbb{Z}^d_+} \lambda^n (\overline{\lambda}')^n = \frac{1}{1 - \langle \lambda, \lambda' \rangle}.$$

In this case the Fourier transform map $\Phi \colon f \mapsto f^\wedge$ has a kernel with the cokernel given by the symmetric Fock space spanned by symmetric tensors

$$\left\{ \sum_{\alpha \in \mathcal{F}_d \colon |\alpha| = n} \{\delta_{\alpha, \alpha'}\}_{\alpha' \in \mathcal{F}_d} e \colon e \in \mathcal{E} \right\}$$

where $\delta_{\alpha,\alpha'}$ is the standard Kronecker delta

$$\delta_{\alpha,\alpha'} = \begin{cases} 1 & \text{if } \alpha = \alpha', \\ 0 & \text{otherwise.} \end{cases}$$

Then it is known (see [22, 6, 10]) that the image of Φ in this case, i.e., the space $H^2(E, \sigma)$ of all functions on the ball of the form $f^\wedge$ for an $f \in \ell^2_{\mathcal{E}}(\mathcal{F}_d)$, is exactly the Arveson-Drury space and the associated space $H^\infty(E, \sigma)$ is exactly the multiplier space $\mathcal{M}(\mathcal{E})$ of the Arveson space. When we specialize the general Theorem 5.1 to this case we simply recover Theorem 2.1 for the case $\mathcal{U} = \mathcal{Y} = \mathcal{E}$.

Case 2. *$\mathcal{G} = \mathbb{C}$ and $\mathcal{E} = \mathcal{E}_0$ is a separable, infinite-dimensional Hilbert space:* In this case the generalized unit disk $\mathbb{D}((E^\sigma)^*) = \mathbb{B}(\mathcal{L}(\mathcal{E}^d, \mathcal{E}))$ consists of row contractions

$$\eta = \begin{bmatrix} \eta_1 & \cdots & \eta_d \end{bmatrix} : \mathcal{E}^d \to \mathcal{E}.$$

The Fock correspondence $\mathcal{F}^2(E) = \ell^2(\mathcal{F}_d)$ has scalar coefficients while the Hilbert Fock space $\mathcal{F}^2(E,\sigma) = \ell^2_{\mathcal{E}}(\mathcal{F}_d)$ has $\mathcal{E}$-valued coefficients. The point-evaluation map (6.6) has the form

$$f = \{f_\alpha\}_{\alpha \in \mathcal{F}_d} \mapsto f^\wedge(\eta, b) = \sum_{\alpha \in \mathcal{F}_d} \eta^\alpha b f_\alpha.$$

The completely positive kernel associated with $H^2(E,\sigma)$ for this case is

$$\mathbb{K}_{E,\sigma}(\eta,\zeta)[b] = \sum_{\alpha \in \mathcal{F}_d} \eta^\alpha [b] \zeta^{\alpha *}.$$

where $b \in \sigma(\mathcal{A})' = \mathcal{L}(\mathcal{E})$.

The analytic Toeplitz algebra $\mathcal{F}^\infty(E)$ is the free semigroup algebra $\mathcal{L}_d$ acting on $\mathcal{F}^2(E) = \ell^2(\mathcal{F}_d)$ having noncommutative Toeplitz matrix representation

$$R = [R_{\alpha,\beta}]_{\alpha,\beta \in \mathcal{F}_d} \quad \text{where} \quad R_{\alpha,\beta} = R_{\alpha\beta^{-1},\emptyset}.$$

where the matrix entries $R_{\alpha,\beta}$ are scalars. Here $\emptyset$ refers to the empty word in $\mathcal{F}_d$ (the unit element for the semigroup $\mathcal{F}_d$) and we use the convention

$$\alpha\beta^{-1} = \begin{cases} \alpha' & \text{if } \alpha = \alpha'\beta, \\ \text{undefined} & \text{otherwise,} \end{cases}$$
$$R_{\text{undefined}} = 0.$$

Then it is easily seen that $R \otimes_\sigma I_{\mathcal{E}} \in \mathcal{L}(\ell^2_{\mathcal{E}}(\mathcal{F}_d))$ is simply the infinite-multiplicity inflation of R:

$$R \otimes I_{\mathcal{E}} = [R \otimes I_{\mathcal{E}}]_{\alpha,\beta} \quad \text{where} \quad [R \otimes I_{\mathcal{E}}]_{\alpha,\beta} = R_{\alpha,\beta} \otimes I_{\mathcal{E}}.$$

The point-evaluation $\eta \mapsto (R \otimes_\sigma I_{\mathcal{E}})^\wedge(\eta)$ for $R \otimes_\sigma I_{\mathcal{E}} \in \mathcal{F}^\infty(E,\sigma)$ and $\eta = \begin{bmatrix} \eta_1 & \cdots & \eta_d \end{bmatrix} \in \mathbb{B}(\mathcal{L}(\mathcal{E}^d, \mathcal{E}))$ is given by

$$(R \otimes_\sigma I_{\mathcal{E}})^\wedge(\eta) = \sum_{\alpha \in \mathcal{F}_d} \eta^\alpha (R_\alpha \otimes I_{\mathcal{E}}).$$

Viewing the operator $R_\alpha \otimes_\sigma I_{\mathcal{E}}$ as simply multiplication by the scalar R_α, we can rewrite this as

$$(R \otimes_\sigma I_{\mathcal{E}})^\wedge(\eta) = \sum_{\alpha \in \mathcal{F}_d} R_\alpha \eta^\alpha. \tag{6.8}$$

As a consequence of the fact that there are no polynomial identities valid for matrices of all sizes (see [44, pp. 22–23]), it follows that the point-evaluation map

$$R \in \mathcal{F}^\infty(E) = \mathcal{L}_d \mapsto \big(\eta \in \mathbb{B}(\mathcal{L}(\mathcal{E}^d, \mathcal{E}) \mapsto (R \otimes_\sigma I_{\mathcal{E}})^\wedge(\eta) \in \mathcal{L}(\mathcal{E})\big)$$

is injective.

For π a $*$-representation of $\mathcal{L}(\mathcal{E})$ into $\mathcal{L}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$, one can check that

$$\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_d \end{bmatrix} \otimes h \cong \begin{bmatrix} \pi(\mu_1)h \\ \vdots \\ \pi(\mu_d)h \end{bmatrix} \tag{6.9}$$

gives an identification of $E^\sigma \otimes \mathcal{H}$ with $\mathcal{H}^d$. For a colligation $\mathbf{U}$ to be of the form (5.3) and to satisfy (5.4) means that there is a Hilbert space $\mathcal{H}$ together with a $*$-representation $\pi\colon \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{H})$ such that, after the identification of $E^\sigma \otimes_\pi \mathcal{H}$ with $\mathcal{H}^d$ via (6.9),

$$\mathbf{U} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \vdots \\ \mathcal{H} \\ \mathcal{E} \end{bmatrix} \tag{6.10}$$

subject to

$$\begin{bmatrix} \pi(b) & & & \\ & \ddots & & \\ & & \pi(b) & \\ & & & b \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} \begin{bmatrix} \pi(b) & 0 \\ 0 & b \end{bmatrix},$$

or, equivalently,

$$\begin{aligned} &A_j\pi(b) = \pi(b)A_j, \qquad B_j b = \pi(b)B_j \text{ for } j = 1, \dots, d, \\ &C\pi(b) = bC, \qquad Db = bD, \end{aligned} \tag{6.11}$$

for all $b \in \mathcal{L}(\mathcal{E})$. For $\eta = \begin{bmatrix} \eta_1 & \cdots & \eta_d \end{bmatrix} \in \mathbb{D}((E^\sigma)^*) = \mathbb{B}(\mathcal{L}(\mathcal{E}^d, \mathcal{E}))$, one can check that the operator $L_{\eta^*} : \mathcal{H} \to E^\sigma \otimes_\pi \mathcal{H}$ given by $L_{\eta^*} : h \mapsto \eta^* \otimes h$, after the identification (6.9), is simply the column contraction

$$L_{\eta^*} : h \mapsto \begin{bmatrix} \eta_1^* \\ \vdots \\ \eta_d^* \end{bmatrix} h$$

with adjoint equal to

$$L_{\eta^*}^* = \begin{bmatrix} \eta_1 & \cdots & \eta_d \end{bmatrix} : \mathcal{H}^d \to \mathcal{H}.$$

Suppose that $S \in H^\infty(E, \sigma)$ for this example of (E, σ). Then the realization formula for $S \in \mathcal{F}^\infty(E, \sigma)$ given by (5.5) for this case becomes

$$S(\eta) = D + C(I - \eta A)^{-1}\eta B \text{ for } \eta = \begin{bmatrix} \eta_1 & \cdots & \eta_d \end{bmatrix} \in \mathbb{B}(\mathcal{L}(\mathcal{E}^d, \mathcal{E})) \tag{6.12}$$

where the coisometric $\mathbf{U}$ is as in (6.10). Using the relations (6.11) and using the expansion

$$(I - \eta A)^{-1} = \sum_{n=0}^{\infty} (\eta A)^n,$$

we see that (6.12) can be rewritten as

$$S(\eta) = D + \sum_{\alpha \in \mathcal{F}_d} \sum_{j=1}^{d} CA^{\alpha} B \eta^{\alpha} \eta_j.$$

Moreover, again from the relations (6.11) we see that

$$(CA^{\alpha}B_j)b = b(CA^{\alpha}B_j) \quad \text{and} \quad Db = bD \text{ for all } b \in \mathcal{L}(\mathcal{E}),$$

i.e., $CA^{\alpha}B_j =: s_{\alpha j}$ *for all* $\alpha \in \mathcal{F}_d$ *and* $j = 1, \dots, d$ *as well as* $D =: s^0_{\emptyset}$ *are all scalar operators:*

$$s_{\alpha} = s^0_{\alpha} I_{\mathcal{E}} \quad \text{where} \quad s^0_{\alpha} \in \mathbb{C}.$$

From the complete positive kernel condition in Theorem 5.1, it is easily seen that $S(\eta) = \sum_{\alpha \in \mathcal{F}_d} s_{\alpha} \eta^{\alpha}$ is contractive for each row contraction $\eta = \begin{bmatrix} \eta_1 & \cdots & \eta_d \end{bmatrix} \in \mathbb{B}(\mathcal{L}(\mathcal{E}^d, \mathcal{E}))$. Thus the formal power series

$$S^0(z) = \sum_{\alpha \in \mathcal{F}_d} s^0_{\alpha} z^{\alpha}$$

is in the formal noncommutative Schur class with scalar coefficients $\mathcal{S}_{nc,d}(\mathbb{C}, \mathbb{C})$ introduced in Section 2.2.

Conversely, if $S^0(z) = \sum_{\alpha \in \mathcal{F}_d} s^0_{\alpha} z^{\alpha}$ is in the formal noncommutative Schur class $\mathcal{S}_{nc,d}(\mathbb{C}, \mathbb{C})$, then part (3) of Theorem 2.3 assures us that $S(z)$ has a realization of the form

$$S^0(z) = D^0 + C^0(I - Z(z)A^0)^{-1} Z(z) B^0 \tag{6.13}$$

for a coisometric (even unitary) colligation

$$\mathbf{U}^0 = \begin{bmatrix} A^0_1 & B^0_1 \\ \vdots & \vdots \\ A^0_d & B^0_d \\ C^0 & D^0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}^0 \\ \mathbb{C} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}^0 \\ \vdots \\ \mathcal{H}^0 \\ \mathbb{C} \end{bmatrix}.$$

Let us form a new tensored colligation $\mathbf{U}$ of the form

$$\mathbf{U} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \vdots \\ \mathcal{H} \\ \mathcal{E} \end{bmatrix}$$

where we set

$$\mathcal{H} = \mathcal{H}^0 \otimes \mathcal{E}, \quad A_j = A^0_j \otimes I_{\mathcal{E}}, \qquad B_j = B^0_j \otimes I_{\mathcal{E}}, \qquad C = C^0 \otimes I_{\mathcal{E}}, \qquad D = D^0 \otimes I_{\mathcal{E}}$$

where $j = 1, \dots, d$. We may define a $*$-representation $\pi \colon \mathcal{L}(\mathcal{E}) \to \mathcal{L}(\mathcal{H})$ by

$$\pi(b) = I_{\mathcal{H}^0} \otimes b.$$

Then it is easily seen that this $\mathbf{U}$ satisfies (6.10) and (6.11). Moreover, from these relations and the realization (6.13) for the formal noncommutative Schur-class function $S^0(z)$, we see that we have a realization for the associated function $\eta \mapsto S^0(\eta)$ of the form (6.12):

$$S(\eta) := \sum_{\alpha \in \mathcal{F}_d} s^0_\alpha \eta^\alpha = D + C(I - L^*_{\eta^*} A)^{-1} L^*_{\eta^*} B.$$

We conclude: *there is a one-to-one correspondence between formal power series*

$$S^0(z) = \sum_{\alpha \in \mathcal{F}_d} s^0_\alpha z^\alpha$$

in the noncommutative scalar-coefficient Schur class $\mathcal{S}_{nc,d}(\mathbb{C}, \mathbb{C})$ *and functions* $\eta \mapsto S(\eta)$ *in the Muhly-Solel class* $H^\infty(E, \sigma)$ *for the particular choice of* (E, σ) (described in (6.5) with $\mathcal{G} = \mathbb{C}$ and $\mathcal{E}_0 = \mathcal{E}$ infinite-dimensional), *given by*

$$S^0(z) = \sum_{\alpha \in \mathcal{F}_d} s^0_\alpha z^\alpha \mapsto \left(\eta \mapsto S(\eta) := \sum_{\alpha \in \mathcal{F}_d} \eta^\alpha (s^0_\alpha I_{\mathcal{E}}) \right).$$

Here we have made explicit the correspondence between condition (3) in Theorem 2.3 for $S^0(z)$ versus condition (3) in Theorem 5.1 for $S(\eta)$. An amusing exercise would be to understand directly the equivalence between any of the other conditions in Theorem 2.3 for $S^0(z)$ and the corresponding condition for $S(\eta)$ in Theorem 5.1.

6.3. Analytic crossed-product algebras

We discuss here a particular case of analytic crossed-product algebras (see Example 2.6 in [31] as well as the references there). This particular case has strong connections with time-varying system theory and was discussed in connection with point-evaluation and generalized Nevanlinna-Pick interpolation in [35] (see Examples 2.5, 2.6 and 2.25 there). Here we wish to draw out the connections between the realization theorem (the equivalence of (1) and (3) in Theorem 5.1 for this case) and a result from [4] that any lower-triangular contractive operator on $\ell^2(\mathbb{Z})$ can be realized as the input-output map of a linear time-varying input/state/output system. For simplicity we discuss in detail only the multiplicity-free case ($\mathcal{U} = \mathcal{Y} = \mathbb{C}$).

We take the algebra $\mathcal{A}$ to be the algebra $\ell^\infty(\mathbb{Z})$ with coordinate-wise multiplication with correspondence E equal to $\mathcal{A} = \ell^\infty(\mathbb{Z})$ as a set. Let α be the automorphisms $\alpha(a)(k) = a(k-1)$ ($k \in \mathbb{Z}$) for $a \colon \mathbb{Z} \to \mathbb{C}$ in $\mathcal{A}$. We consider E as a correspondence over $\mathcal{A}$ with left and right action given by

$$(a \cdot \xi) = (\alpha(a))(k) = a(k-1)\xi(k), \qquad (\xi \cdot a)(k) = (\xi a)(k) = \xi(k)a(k)$$

and with the $\mathcal{A}$-valued inner product

$$\langle \xi', \xi \rangle_E(k) = \overline{\xi(k)} \xi'(k) \tag{6.14}$$

for $d \in \mathbb{Z}$, $a \in \mathcal{A} = \ell^\infty(\mathbb{Z})$ and $\xi', \xi \in E = \ell^\infty(\mathbb{Z})$. Then it is easily seen that $E^{\otimes n}$ is the correspondence over $\mathcal{A}$ identified again with $E = \ell^\infty(\mathbb{Z})$ as a set with

$\mathcal{A}$-valued inner product as in (6.14) but with left and right $\mathcal{A}$-action given by

$$(a\cdot\xi^{(n)})(k) = (\alpha^n(a)\xi^{(n)})(k) = a(k-n)\xi(k),$$
$$(\xi^{(n)}\cdot a)(k) = (\xi^{(n)}a)(k) = \xi^{(n)}(k)a(k)$$

for $k\in\mathbb{Z}$, $\xi^{(n)}\in E^{\otimes n}=\ell^\infty(\mathbb{Z})$ and $a\in\mathcal{A}=\ell^\infty(\mathbb{Z})$. The Fock space $\mathcal{F}^2(E)$ is then the correspondence $\oplus_{n=0}^\infty\ell^\infty(\mathbb{Z})$ with left and right $\mathcal{A}$-action given by

$$a\cdot(\oplus_{n=0}^\infty\xi^{(n)}) = \oplus_{n=0}^\infty\alpha^n(a)\xi^{(n)}, \qquad (\oplus_{n=0}^\infty\xi^{(n)})\cdot a = \oplus_{n=0}^\infty\xi^{(n)}a.$$

More generally, when $\mathcal{A}$ is a general von Neumann algebra and α is an automorphism of $\mathcal{A}$, this construction gives rise to analytic crossed-product algebras which have been studied by a number of authors over the past several decades (see [31, 33, 35] and the references therein).

An appealing alternative representation of the correspondence, as explained in Example 2.6 in [35], is as follows. View $\mathcal{A}$ as the algebra $\mathcal{D}$ of all diagonal operators acting on $\ell^2(\mathbb{Z})$, let U be the bilateral shift operator $Ue_k=e_{k+1}$ (where $e_k(k')$ is the Kronecker delta function) and let $E=U\mathcal{D}\subset\mathcal{L}(\ell^2(\mathbb{Z}))$. Then define the left and right actions of $\mathcal{A}=\mathcal{D}$ on $E=U\mathcal{D}$ simply by left and right operator multiplications with the inner product given by

$$\langle UD_1, UD_2\rangle_{\mathcal{E}} = D_2^*D_1\in\mathcal{D}.$$

One can check that this $\mathcal{A}$-correspondence E is unitarily equivalent to the $\mathcal{A}'$-correspondence E' given in the previous paragraph with the obvious identifications:

$$\{d(k)\}_{k\in\mathbb{Z}}\in\mathcal{A}'=\ell^\infty(\mathbb{Z})\mapsto \mathrm{diag}_{k\in\mathbb{Z}}d(k)\in\mathcal{D}=\mathcal{A},$$
$$\{\xi(k)\}_{k\in\mathbb{Z}}\in E'=\ell^\infty(\mathbb{Z})\mapsto U\cdot(\mathrm{diag}_{k\in\mathbb{Z}}\xi(k))\in U\mathcal{D}=E.$$

One can easily check that the identification map

$$UD_2\otimes UD_1\in U\mathcal{D}\otimes U\mathcal{D}\mapsto UD_2UD_1 = U^2(U^*D_2U)D_1\in U^2\mathcal{D}.$$

is unitary from $E\otimes E$ to $U^2\mathcal{D}$. After this identification, the left and right $\mathcal{D}$-action on $E^{\otimes 2}\cong U^2\mathcal{D}$ is again given by left and right operator multiplication. More generally, we view $E^{\otimes n}$ as $U^n\mathcal{D}$ with left and right $\mathcal{D}$-action given by operator multiplication and with inner product inherited from $\mathcal{L}(\ell^2(\mathbb{Z}))$:

$$\langle U^nD_1, U^nD_2\rangle_{E^{\otimes n}} = D_2^*U^{n*}U^nD_1 = D_2^*D_1\in\mathcal{D}.$$

The Fock space $\mathcal{F}^2(E)$ can then be identified with lower triangular matrices T with diagonal expansion $T=\sum_{n=0}^\infty U^nD_n$ ($D_n\in\mathcal{D}$) such that

$$\sum_{n=0}^N D_n^*D_n \text{ is bounded above in } \mathcal{D}. \tag{6.15}$$

The Toeplitz algebra $\mathcal{F}^\infty(E)$ consists of all lower triangular matrices R which give rise to bounded operators on $\ell^2(\mathbb{Z})$. As elements of $\mathcal{F}^\infty(E)$, they act on $\mathcal{F}^2(E)$ (lower triangular matrices satisfying (6.15)) via multiplication on the left. We can view this algebra as generated by a single creation operator T_I (the creation operator associated with the identity matrix $I\in\mathcal{D}$, namely the bilateral shift operator U), together with the diagonal operators $\mathcal{D}$. Note that U is really a unilateral shift operator since it is restricted to the space $\mathcal{F}^2(E)$ of lower triangular matrices (with action equal to a shifting of the subdiagonals).

We now set $\mathcal{E} = \ell^2(\mathbb{Z})$ and let σ be the identity representation of $\mathcal{D}$ on $\mathcal{E} = \ell^2(\mathbb{Z})$. Then $E^{\otimes n} \otimes_\sigma \mathcal{E}$ can be identified with $\mathcal{E} = \ell^2(\mathbb{Z})$ in the natural way

$$\iota\colon U^n D \otimes e \mapsto De.$$

When this is done the left action of $\mathcal{A} = \mathcal{D}$ becomes

$$d \cdot e = U^{n*} d U^n e$$

since

$$\begin{aligned}\iota(d \cdot (U^n D \otimes e)) &= \iota(dU^n D \otimes e) = \iota(U^n(U^{n*}dU^n)D \otimes e)\\ &= U^{n*}dU^n De = U^{n*}dU^n \iota(U^n D \otimes e).\end{aligned}$$

Hence we identify $\mathcal{F}^2(E, \sigma) = \mathcal{F}^2(E) \otimes_\sigma \mathcal{E}$ with

$$\mathcal{F}^2(E, \sigma) = \ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+)$$

with left action by $\mathcal{A} = \mathcal{D}$ given by

$$b \cdot \{e_n\}_{n\in\mathbb{Z}_+} = \{U^{*n} b U^n e_n\}_{n\in\mathbb{Z}_+}.$$

One can see that the image of the generating creation operator $\mathbf{T}_I = U \otimes I_{\mathcal{E}}$ after these identifications is the unilateral shift operator $S \otimes I_{\ell^2(\mathbb{Z})}$ acting on $\mathcal{F}^2(E, \sigma) = \ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+)$:

$$\mathbf{T}_I = [t_{i,j}]_{i,j\in\mathbb{Z}_+} \quad \text{where} \quad t_{i,j} = \begin{cases} I_{\ell^2(\mathbb{Z})} & \text{if } i = j+1,\\ 0 & \text{otherwise.}\end{cases}$$

The elements R of $\mathcal{F}^\infty(E, \sigma) \subset \mathcal{L}(\ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+))$ can then be identified as the following algebra of sparse matrices: there is a sequence $\{d_n\}_{n\in\mathbb{Z}} \subset \mathcal{D}$ of diagonal operators on $\ell^2(\mathbb{Z})$ so that R has the form

$$R = [R_{i,j}]_{i,j\in\mathbb{Z}_+} \quad \text{where} \quad R_{i,j} = \begin{cases} U^{*j} d_{i-j} U^j & \text{for } i \geq j,\\ 0 & \text{otherwise,}\end{cases} \tag{6.16}$$

or, in block-matrix form,

$$R = \begin{bmatrix} d_0 & 0 & 0 & \cdots \\ d_1 & U^* d_0 U & 0 & \cdots \\ d_2 & U^* d_1 U & U^{*2} d_0 U^2 & \cdots \\ \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

We identify $(E^\sigma)^*$ for this example as follows. The space $(E^\sigma)^*$ consists of operators $\eta\colon E \otimes_\sigma \mathcal{E} \to \mathcal{E}$ such that $\eta(\varphi(a) \otimes I_{\mathcal{E}}) = \sigma(a)\eta$. For the present situation, both $E \otimes_\sigma \mathcal{E}$ and $\mathcal{E}$ are identified with $\ell^2(\mathbb{Z})$ but the left action by an element $d \in$

$\mathcal{A} = \mathcal{D}$ is given by multiplication by U^*dU in the first case and by multiplication by d in the second. Thus the operator $\eta \in \mathcal{L}(\ell^2(\mathbb{Z}))$ is required to satisfy

$$\eta U^* D U = D\eta$$

which means that ηU^* is diagonal, so $(E^\sigma)^*$ is identified with weighted shift operators

$$(E^\sigma)^* \cong \{\eta = D_\eta U = U(U^* D_\eta U) \in \mathcal{L}(\ell^2(\mathbb{Z}))\colon D_\eta \in \mathcal{D}\} = U\mathcal{D}. \tag{6.17}$$

Recall that there is a representation of $\mathcal{F}^\infty(E^\sigma)$ on $\mathcal{F}^2(E,\sigma)$ (where E^σ is viewed as a $\sigma(\mathcal{A})'$-correspondence). For our situation here, $\sigma(\mathcal{A})' = \mathcal{A} = \mathcal{D}$ considered as acting on $\mathcal{E} = \ell^2(\mathbb{Z})$ and the representation of $\sigma(\mathcal{A})'$ on $\mathcal{F}(E,\sigma) = \ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+)$ turns out to be the diagonal action:

$$b \cdot (\oplus_{n=0}^\infty e_n) = \oplus_{n=0}^\infty b e_n \text{ for } b \in \mathcal{D},\ \oplus_{n=0}^\infty e_n \in \ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+). \tag{6.18}$$

For purposes of getting a generating set for $\mathcal{F}^\infty(E^\sigma)$, it suffices to consider the single creation operator associated with $\eta^* = U^*$: the associated action on $\mathcal{F}^2(E,\sigma)$ turns out to be

$$T^d_{U^*,\sigma} = [t'_{i,j}] \quad \text{where} \quad t'_{i,j} = \begin{cases} U^* & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.19}$$

According to the duality result from [33], an operator R on $\ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+)$ is of the form (6.16) if and only if R commutes with the scalar-diagonal operators (6.18) and the E^σ-creation operator (6.19); an amusing exercise for the reader is to verify this fact directly for this example.

We now identify the Z-transform and compute the function spaces $H^2(E,\sigma)$ and $H^\infty(E,\sigma)$ as follows. By (6.17) we have an identification of $(E^\sigma)^*$ with the space of weighted shift operators $U\mathcal{D}$ in $\mathcal{L}(\ell^2(\mathbb{Z}))$. After carrying out the identifications $E^{\otimes n} \otimes_\sigma \mathcal{E} = \ell^2(\mathbb{Z})$, one can check that the generalized power $\eta^n\colon E^{\otimes n} \otimes_\sigma \mathcal{E} \to \mathcal{E}$ of an $\eta \in (E^\sigma)^* = U\mathcal{D}$ coincides with the usual power η^n as an element of the operator algebra $\mathcal{L}(\ell^2(\mathbb{Z}))$. Therefore, for $f = \{f_n\}_{n\in\mathbb{Z}_+} \in \ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+)$ and $\eta = D_\eta U \in \mathbb{D}((E^\sigma)^*)$ (with $D_\eta \in \mathcal{D}$), we have

$$f^\wedge(\eta, b) = \sum_{n=0}^\infty \eta^n b f_n = \sum_{n=0}^\infty (D_\eta U)^n b f_n.$$

If we restrict the second variable $b \in \sigma(\mathcal{A})' = \mathcal{D}$ to be $b = I_{\ell^2(\mathbb{Z})}$, we have the restricted Fourier transform

$$\Phi^1\colon f = \{f_n\}_{n\in\mathbb{Z}_+} \mapsto f^\wedge(\eta, I_{\ell^2(\mathbb{Z})}) = \sum_{n=0}^\infty \eta^n f_n = \sum_{n=0}^\infty (D_\eta U)^n f_n.$$

We assert that *the restricted Z-transform $\Phi^1\colon f \mapsto f^{\wedge 1} := f^\wedge(\cdot, I_{\ell^2(\mathbb{Z})})$ is injective.* Indeed, if $f^{\wedge 1}(\eta) = 0$ for all η, evaluating at $\eta = 0$ gives that $f_0 = 0$ and hence

$F^{\wedge 1}(\eta) = \eta \cdot \sum_{n=0}^{\infty} f_{n+1}\eta^n = 0$. Choosing η invertible and premultiplying by η^{-1} then gives that

$$\sum_{n=0}^{\infty} f_{n+1}\eta^n = 0 \tag{6.20}$$

for all invertible η. By approximating a noninvertible η by invertible η's, we see that (6.20) actually holds for all $\eta \in \mathbb{D}((E^\sigma)^*)$. Iteration of the same argument now gives that $f_n = 0$ for all $n \in \mathbb{Z}_+$, i.e., $f = \{f_n\}_{n\in\mathbb{Z}_+}$ is the zero element of $\ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+)$, and the assertion follows. Note that the $\sigma(\mathcal{A})' = \mathcal{D}$-action on $\mathcal{F}^2(E,\sigma)$ is given by

$$d \cdot \{f_n\}_{n\in\mathbb{Z}_+} = \{df_n\}_{n\in\mathbb{Z}_+} \text{ for } d \in \mathcal{D}.$$

The completely positive kernel $\mathbb{K}$ associated with the reproducing kernel Hilbert correspondence $H^2(E,\sigma) = \Phi(\ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+))$ is

$$\mathbb{K}(\eta,\zeta)[b] = \sum_{n=0}^{\infty} \eta^n b \zeta^{*n} \text{ for } \eta,\zeta \in \mathbb{D}((E^\sigma)^*) = \mathbb{B}(U\mathcal{D}) \quad \text{and} \quad b \in \mathcal{D}.$$

Note that $\Phi^*\mathbb{K}(\cdot,\zeta)[b]e = bk_\zeta e$ where

$$bk_\eta e = \{b\zeta^{*n}e\}_{n\in\mathbb{Z}_+} \in \ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+).$$

We conclude that the subcollection

$$\{bk_\zeta e \colon b \in \mathcal{D},\ \zeta \in \mathbb{B}(U\mathcal{D}),\ e \in \ell^2(\mathbb{Z})\}$$

has dense span in $\ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+)$.

An element R of $\mathcal{F}^\infty(E)$ is identified with a lower triangular matrix representing a bounded operator on $\ell^2(\mathbb{Z})$; it is convenient to represent such a matrix via a generalized Fourier series along subdiagonals:

$$R \sim \sum_{n=0}^{\infty} U^n d_n \quad \text{where} \quad d_n \in \mathcal{D}. \tag{6.21}$$

(The Caesaro averages of the partial sums of the series converges to R in the weak-$*$ topology but we shall not need this.) Then $R \otimes I_{\ell^2(\mathbb{Z})}$, after the identification of $\mathcal{F}^2(E) \otimes_\sigma \ell^2(\mathbb{Z})$ with $\ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+)$, is identified with the operator acting on $\ell^2_{\ell^2(\mathbb{Z})}(\mathbb{Z}_+)$ with the sparse matrix representation (6.16). For $\eta \in \mathbb{D}((E^\sigma)^*) = \mathbb{B}(U\mathcal{D})$, the associated point evaluation of $R \otimes I_{\mathcal{E}}$ is then given by

$$\begin{aligned}(R \otimes I_{\ell^2(\mathbb{Z})})^\wedge(\eta) &= \sum_{n=0}^{\infty} \eta^n R_{n,0} \\ &= \sum_{n=0}^{\infty} \eta^n d_n \end{aligned} \tag{6.22}$$

if R is given by (6.21). In particular, formally we recover R from $(R \otimes I_{\ell^2(\mathbb{Z})})^\wedge$ as

$$R = (R \otimes I_{\ell^2(\mathbb{Z})})^\wedge(U). \tag{6.23}$$

More precisely, we interpret the right-hand side of (6.23) as

$$(R \otimes I_{\ell^2(\mathbb{Z})})^\wedge(U) = \lim_{r\uparrow 1}(R \otimes I_{\ell^2(\mathbb{Z})})^\wedge(rU).$$

The realization theorem (the equivalence of (1) and (3) in Theorem 5.1) assures us that any function of the form $(R \otimes I_{\ell^2(\mathbb{Z})})^\wedge$ with $\|R\| \le 1$ can be realized as follows. Suppose first that $\mathcal{H}$ is a $(\sigma(\mathcal{A})' = \mathcal{D}, \mathbb{C})$-correspondence, i.e., $\mathcal{H}$ is a Hilbert space and there is a $*$-representation π of $\sigma(\mathcal{A})' = \mathcal{D}$ with values in $\mathcal{L}(\mathcal{H})$. Noting that

$$U^* d \otimes_\pi h = U^* \otimes_\pi \pi(d)h \cong \pi(d)h$$

for $\mu = U^*d \in E^\sigma = U^*\mathcal{D}$ (so $d \in \mathcal{D} = \sigma(\mathcal{A})'$) and $h \in \mathcal{H}$, we see that $E^\sigma \otimes_\pi \mathcal{H}$ can be identified with $\mathcal{H}$, but at the price that the left $(\sigma(\mathcal{A})' = \mathcal{D})$-action on $\mathcal{H}$ is given by $\pi^{(1)} \colon b \mapsto \pi(UbU^*)$ rather than by π. With this identification, we see that the unitary colligation $\mathbf{U}$ in (5.3) and (5.4) for this case has the form

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \ell^2(\mathbb{Z}) \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \ell^2(\mathbb{Z}) \end{bmatrix}$$

subject to

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \pi(b) & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} \pi^{(1)}(b) & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ for all } b \in \sigma(\mathcal{A})' = \mathcal{D},$$

or, equivalently,

$$A\pi(b) = \pi(UbU^*)A, \quad Bb = \pi(UbU^*)B, \quad C\pi(b) = bC, \quad Db = bD \tag{6.24}$$

for all $b \in \mathcal{D}$. The realization theorem then tells us that any $(R \otimes I_{\ell^2(\mathbb{Z})})^\wedge$ (where $R \in \mathcal{L}(\ell^2(\mathbb{Z}))$ is lower-triangular and contractive) can be realized as

$$(R \otimes I_{\ell^2(\mathbb{Z})})^\wedge(\eta) = D + C(I - \pi(\eta U^*)A)^{-1}\pi(\eta U^*)B. \tag{6.25}$$

Let us now consider a time-varying input/state/output linear system of the form

$$\Sigma : \begin{cases} x(n+1) &= A(n)x(n) + B(n)u(n) \\ y(n) &= C(n)x(n) + D(n)u(n). \end{cases} \tag{6.26}$$

determined by the time-varying system matrix

$$\mathbf{U}(n) = \begin{bmatrix} A(n) & B(n) \\ C(n) & D(n) \end{bmatrix} : \begin{bmatrix} \mathcal{H}(n) \\ \mathbb{C} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(n+1) \\ \mathbb{C} \end{bmatrix}.$$

We say that the system is *conservative* (respectively, *dissipative*) if each $U(n)$ is unitary (respectively, contractive). Let us assume that we have a dissipative time-varying linear system with time-varying system matrix $\mathbf{U}(n) = \left[\begin{smallmatrix} A(n) & B(n) \\ C(n) & D(n) \end{smallmatrix}\right]$. Then it can be shown that, given an input string $\{u(n)\}_{n\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$, there is a unique system trajectory $(u(n), x(n), y(n))$, i.e., solution of the system equations (6.26), such that $\lim_{n\to-\infty} x(n) = 0$ with the resulting output string $\{y(n)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. In this way there is defined an input-output map T_Σ on $\ell^2(\mathbb{Z})$ such that $T_\Sigma \colon \{u(n)\}_{n\in\mathbb{Z}} \mapsto \{y(n)\}_{n\in\mathbb{Z}}$.

Let us introduce an aggregate state space

$$\mathcal{H} = \oplus_{n\in\mathbb{Z}}\mathcal{H}(n) \tag{6.27}$$

and an aggregate system matrix

$$\mathbf{U} = \begin{bmatrix} \boldsymbol{\mathcal{A}} & \boldsymbol{\mathcal{B}} \\ \boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{D}} \end{bmatrix} \begin{bmatrix} \mathcal{H} \\ \ell^2(\mathbb{Z}) \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \ell^2(\mathbb{Z}) \end{bmatrix} \tag{6.28}$$

with $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{D}}$ specified by block-matrix entries

$$\begin{aligned} [\boldsymbol{\mathcal{A}}]_{i,j} &= A(j)\delta_{i,j+1}, \qquad [\boldsymbol{\mathcal{B}}]_{i,j} = B(j)\delta_{i,j+1}, \\ [\boldsymbol{\mathcal{C}}]_{i,j} &= C(j)\delta_{i,j}, \qquad [\boldsymbol{\mathcal{D}}]_{i,j} = D(j)\delta_{i,j}. \end{aligned} \tag{6.29}$$

If the operator $\boldsymbol{\mathcal{A}}$ has spectral radius strictly less than 1 as an operator on $\mathcal{H}$, then one can compute that T_Σ is given by

$$T_\Sigma = \boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{C}}(I - \boldsymbol{\mathcal{A}})^{-1}\boldsymbol{\mathcal{B}} \in \mathcal{L}(\ell^2(\mathbb{Z})). \tag{6.30}$$

Even if $\boldsymbol{\mathcal{A}}$ does not have spectral radius strictly less than 1, there are various ways whereby one can still make sense of the formula (6.30); one such is via a limit

$$T_\Sigma = \lim_{r\uparrow 1} \boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{C}}(I - r\boldsymbol{\mathcal{A}}))^{-1}(r\boldsymbol{\mathcal{B}}).$$

From the representation (6.30) for T_Σ one can compute that T_Σ has the diagonal decomposition

$$T_\Sigma = \sum_{n=0}^{\infty} U^n d_n \quad \text{where} \quad d_0 = \boldsymbol{\mathcal{D}} \quad \text{and} \quad d_n = U^{*n}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{A}}^{n-1}\boldsymbol{\mathcal{B}} \text{ for } n \geq 1.$$

Hence an application of (6.22) gives us

$$(T_\Sigma \otimes I_{\ell^2(\mathbb{Z})})^{\wedge}(\eta) = \boldsymbol{\mathcal{D}} + \sum_{n=1}^{\infty} \eta^n U^{*n}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{A}}^{n-1}\boldsymbol{\mathcal{B}}. \tag{6.31}$$

Given $\mathcal{H}$ in the form (6.27), we may define a representation π of $\mathcal{D}$ by

$$\pi(b)\colon\ \oplus_{n\in\mathbb{Z}}\, h(n) \mapsto \oplus_{n\in\mathbb{Z}} b(n)h(n) \text{ for } b = \operatorname{diag}_{n\in\mathbb{Z}}\{b(n)\} \in \mathcal{D}.$$

Note that if $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}}$ are as in (6.29), then $\mathbf{U}$ as in (6.28) satisfies the $\mathcal{D}$-module property (6.24) (with $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}}$ in place of A, B, C, D). By a careful induction argument making using of these relations, one can show that

$$\boldsymbol{\mathcal{C}}(\pi(\eta U^*)\boldsymbol{\mathcal{A}})^{k-1}\pi(\eta U^*) = \eta^k U^{*k}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{A}}^{k-1} \text{ for } k = 1, 2, \ldots.$$

One can then show that

$$\begin{aligned} \boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{C}}(I - \pi(\eta U^*)\boldsymbol{\mathcal{A}})^{-1}\pi(\eta U^*)\boldsymbol{\mathcal{B}} &= \boldsymbol{\mathcal{D}} + \sum_{n=1}^{\infty} \boldsymbol{\mathcal{C}}(\pi(\eta U^*)\boldsymbol{\mathcal{A}})^{n-1}\pi(\eta U^*)\boldsymbol{\mathcal{B}} \\ &= \boldsymbol{\mathcal{D}} + \sum_{n=1}^{\infty} \eta^n U^{*n}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{A}}^{n-1}\boldsymbol{\mathcal{B}} \\ &= (T_\Sigma \otimes I_{\ell^2(\mathbb{Z})})^{\wedge}(\eta), \end{aligned}$$

i.e., *the aggregate colligation* $\mathbf{U} = \left[\begin{smallmatrix} \boldsymbol{\mathcal{A}} & \boldsymbol{\mathcal{B}} \\ \boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{D}} \end{smallmatrix}\right]$ *arising from the realization of* T_Σ *as the input-output map for the time-varying linear system* (6.26) *gives rise to a realization of the form* (6.25) *for the function* $(T_\Sigma \otimes I_{\ell^2(\mathbb{Z})})^\wedge$ *in the Muhly-Solel Schur class for this special setting.*

This suggests a different approach to the realization theorem (the equivalence of (1) and (3) in Theorem 5.1) for this particular case. Given a contractive lower-triangular operator R on $\ell^2(\mathbb{Z})$, it is known (see [4, Theorem 6.2]) that one can realize R as the input-output map $R = T_\Sigma$ of a conservative time-varying input/state/output linear system as in (6.26); the solution in [4] is given via a time-varying analogue of the Pavlov functional model, or, alternatively, via a time-varying analogue of the Sz.-Nagy-Foias or de Branges-Rovnyak functional model. Once we have realized R as $R = T_\Sigma$ with Σ as in (6.26), we get $(R \otimes I_{\ell^2(\mathbb{Z})})^\wedge$ realized in the form (6.25) and hence we have recovered the implication (1) $\Longrightarrow$ (3) of Theorem 5.1. We conclude that the Muhly-Solel realization theorem for this case, after some translation, has essentially the same content as the conservative realization theorem for linear time-varying systems in [4].

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Some Problems Concerning the Test Functions in the Szegö and Avram-Parter Theorems

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Abstract. The Szegö and Avram-Parter theorems give the limit of the arithmetic mean of the values of certain test functions at the eigenvalues and singular values of Toeplitz matrices as the matrix dimension increases to infinity. This paper is concerned with some questions that arise when the test functions do not satisfy the known growth restrictions at infinity or when the test function has a logarithmic singularity within the range of the symbol. Several open problems are listed and accompanied by a few new results that illustrate the delicacy of the matter.

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Keywords. Toeplitz matrix, Toeplitz determinant, test function, singular value, eigenvalue.

1. Introduction

Given an $n \times n$ matrix A, we denote by $s_1(A) \le \cdots \le s_n(A)$ the singular values of A, and if A is Hermitian, $A = A^*$, we let $\lambda_1(A) \le \cdots \le \lambda_n(A)$ stand for the eigenvalues of A. The $n \times n$ Toeplitz matrix generated by a function $a \in L^1(-\pi, \pi)$ is the matrix $T_n(a) := (a_{j-k})_{j,k=1}^n$ where a_j is the jth Fourier coefficient of a,

$$a_j = \int_{-\pi}^{\pi} a(\theta) e^{-ij\theta} \, \frac{d\theta}{2\pi} \quad (j \in \mathbb{Z}).$$

If a is real-valued, then $T_n(a)$ is Hermitian and the (first) Szegö limit theorem states that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j(T_n(a))) = \int_{-\pi}^{\pi} F(a(\theta)) \, \frac{d\theta}{2\pi} \tag{1}$$

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for appropriate functions $F : \mathbb{R} \to \mathbb{R}$. Theorems of the Avram-Parter type concern complex-valued functions a and say that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) = \int_{-\pi}^{\pi} F(|a(\theta)|)\, \frac{d\theta}{2\pi} \tag{2}$$

for certain functions $F : [0, \infty) \to \mathbb{R}$. The functions F in (1) and (2) are referred to as test functions, and the problem consists in proving whether (1) and (2) or a modification of (1) and (2) is true for a given test function F. This paper is devoted to a few open questions pertaining to this problem, and it also contains some new results.

Here are two concrete problems we have been unable to solve. They provide the reader with an idea of the kind of questions considered in this paper.

Problem 1.1. *Let a be a real-valued trigonometric polynomial that assumes both positive and negative values and let $F(\lambda) = \log|\lambda|$ for $\lambda \neq 0$ and $F(0) = 0$. Is there a sequence $n_1 < n_2 < \cdots$ such that*

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} F(\lambda_j(T_{n_k}(a))) = \int_{-\pi}^{\pi} F(a(\theta))\, \frac{d\theta}{2\pi} \ ?$$

Problem 1.2. *Let $F(s) = 0$ for $s \in [0, 1]$ and $F(s) = s \log s$ for $s \in [1, \infty)$. Is (2) true for all $a \in L^1(-\pi, \pi)$ for which the right-hand side of (2) is finite?*

2. Extensions of the Avram-Parter theorem

We abbreviate $L^p(-\pi, \pi)$ to L^p and denote the function $\theta \mapsto F(|a(\theta)|)$ by $F(|a|)$. For simplicity, we assume in this section that $F \geq 0$. The question we are interested in is whether if $a \in L^1$ and $F : [0, \infty) \to [0, \infty)$ is continuous, does it follow that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) = \begin{cases} \int_{-\pi}^{\pi} F(|a(\theta)|)\, \frac{d\theta}{2\pi} & \text{if} \quad F(|a|) \in L^1, \\ \infty & \text{if} \quad F(|a|) \notin L^1. \end{cases}$$

Clearly, this amounts to asking whether

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) = \|F(|a|)\|_1, \tag{3}$$

where $\|\cdot\|_p$ is the norm in L^p, that is, $\|f\|_p^p = \int_{-\pi}^{\pi} |f(\theta)|^p \frac{d\theta}{2\pi}$. It is known that (3) holds if

(a) $a \in L^\infty$ and F is continuous (Parter [8], Avram [1]),
(b) $a \in L^1$ and F is bounded and uniformly continuous (Zamarashkin and Tyrtyshnikov [13], Tilli [12]),
(c) $a \in L^p$ $(1 \leq p < \infty)$ and F is continuous with $F(s) = O(s^p)$ as $s \to \infty$ (Serra Capizzano [10]).

Result (c) implies in particular that (3) is true for all $a \in L^1$ and all continuous $F : [0, \infty) \to [0, \infty)$ satisfying $F(s) = O(s)$. Thus, (b) is contained in (c).

Note also that in all these case $\|F(|a|)\|_1 < \infty$. The following result shows that (3) is always true if $\|F(|a|)\|_1 = \infty$.

Proposition 2.1. *Let $F : [0, \infty) \to [0, \infty)$ be a continuous function and let $a \in L^1$. If*

$$C := \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) < \infty, \tag{4}$$

then $F(|a|) \in L^1$ and $\|F(|a|)\|_1 \le C$.

Proof. Fix $\varepsilon > 0$ and choose $n_1 < n_2 < \cdots$ so that

$$\frac{1}{n_k} \sum_{j=1}^{n_k} F(s_j(T_{n_k}(a))) < C + \varepsilon.$$

For a natural number M, define $F_M : [0, \infty) \to [0, \infty)$ by

$$F_M(s) = \begin{cases} F(s) & \text{for} \quad s \in [0, M], \\ (M + 1 - s)F(s) & \text{for} \quad s \in [M, M + 1], \\ 0 & \text{for} \quad s \in [M + 1, \infty). \end{cases}$$

The function F_M is continuous and has compact support. From (b) we therefore deduce that

$$\begin{aligned} \int_{-\pi}^{\pi} F_M(|a(\theta)|)\, d\theta &= \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} F_M(s_j(T_{n_k}(a))) \\ &\le \limsup_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} F_M(s_j(T_{n_k}(a))) \le C + \varepsilon. \end{aligned} \tag{5}$$

Furthermore,

$$F_1(|a(\theta)|) \le F_2(|a(\theta)|) \le \cdots . \tag{6}$$

By the Beppo Levi theorem, (5) and (6) imply that

$$F(|a(\theta)|) = \lim_{M\to\infty} F_M(|a(\theta)|)$$

is a function in L^1 and that $\|F(|a|)\|_1 \le C + \varepsilon$. □

Corollary 2.2. *If $a \in L^1$, then* (3) *is true whenever $F : [0, \infty) \to [0, \infty)$ is continuous and $F(|a|) \notin L^1$.*

Proof. This is immediate from Proposition 2.1. □

Thus, if (3) would be valid whenever $F : [0, \infty) \to [0, \infty)$ is continuous and $F(|a|) \in L^1$, we could say that the Avram-Parter theorem is true whenever it makes sense. Unfortunately, this is not the case.

Proposition 2.3. *There exist continuous functions $F : [0, \infty) \to [0, \infty)$ and functions $a \in L^1$ such that $F(|a|) \in L^1$ but* (3) *is false.*

Proof. Let $a(\theta) = \theta^{-\alpha}$ for $\theta \in (-\pi, \pi)$. If $0 < \alpha < 1$, then $a \in L^1$. The maximal singular value of the positive definite Hermitian matrix $T_n(a)$ is $\lambda_n = \|T_n(a)\|$. Since $\|T_n(a)\|$ increases monotonically to $\|a\|_\infty = \infty$, there are $n_1 < n_2 < \cdots$ such that $0 < \lambda_{n_1} < \lambda_{n_2} < \cdots$ and $\lambda_{n_k} \to \infty$. Choose $\varepsilon_{n_k} > 0$ small enough and let $F : [0, \infty) \to [0, \infty)$ be the function which increases linearly from 0 to n_k^2 on $[\lambda_{n_k} - \varepsilon_{n_k}, \lambda_{n_k}]$, decreases linearly from n_k^2 to 0 on $[\lambda_{n_k}, \lambda_{n_k} + \varepsilon_{n_k}]$, and is identically zero outside $\cup_{k \geq 1} [\lambda_{n_k} - \varepsilon_{n_k}, \lambda_{n_k} + \varepsilon_{n_k}]$. We then have

$$\frac{1}{n_k} \sum_{j=1}^{n_k} F(s_j(T_{n_k}(a))) \geq \frac{1}{n_k} F(s_{n_k}(T_{n_k}(a))) = \frac{1}{n_k} F(\lambda_{n_k}) = n_k$$

and hence

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) = \infty.$$

On the other hand,

$$\begin{aligned} \|F(|a|)\|_1 &= 2 \int_0^\pi F(\theta^{-\alpha}) \frac{d\theta}{2\pi} = \frac{2}{\alpha} \int_{\pi^{-\alpha}}^\infty F(s) s^{-1/\alpha - 1} ds \\ &\leq \frac{2}{\alpha} \sum n_k^2 (\lambda_{n_k} - \varepsilon_{n_k})^{-1/\alpha - 1} (2\varepsilon_{n_k}) < \infty \end{aligned}$$

if only ε_{n_k} is chosen small enough. Consequently, the right-hand side of (3) is finite, but the limit on the left of (3) does not exist or is infinite. □

The function F constructed in the proof of Proposition 2.3 is not monotonous. This leads to the following question.

Problem 2.4. *Is* (3) *true for every monotonically increasing and continuous function* $F : [0, \infty) \to [0, \infty)$*?*

Corollary 2.2 in conjunction with (c) shows that the answer is in the affirmative if $F(s) \simeq s^p$ $(1 \leq p < \infty)$ as $s \to \infty$, which means that there are constants $0 < C_1 < C_2 < \infty$ such that $C_1 s^p \leq F(s) \leq C_2 s^p$ for all sufficiently large s. Here are some more test functions for which the answer is positive.

Proposition 2.5. *Let* μ *be a nonnegative Borel measure on* $[1, \infty)$ *such that*

$$F(s) := \int_1^\infty s^x \, d\mu(x) < \infty$$

for all $s \geq 0$. *Then* $F : [0, \infty) \to [0, \infty)$ *is a continuous, convex, and monotonically increasing function and* (3) *is true for this* F *and all* $a \in L^1$.

Proof. It is clear that F is nonnegative, monotonically increasing, and convex. This implies that F is continuous. By virtue of Corollary 2.2, it remains to prove (3) under the assumption that

$$\|F(|a|)\|_1 = \int_{-\pi}^{\pi} \int_1^\infty |a(\theta)|^x \, d\mu(x) \, \frac{d\theta}{2\pi} < \infty. \tag{7}$$

Since the iterated integral in (7) is finite, we deduce from Tonelli's theorem that $\int_{-\pi}^{\pi} |a(\theta)|^x \, d\theta < \infty$ for μ-almost all x in the support of the measure μ and that

$$\|F(|a|)\|_1 = \int_1^\infty \int_{-\pi}^{\pi} |a(\theta)|^x \, \frac{d\theta}{2\pi} \, d\mu(x). \tag{8}$$

It follows in particular that $a \in L^x$ for all $x \in \operatorname{supp} \mu$. Avram [1] proved that

$$\frac{1}{n} \sum_{j=1}^{n} s_j(T_n(a))^x \le \|a\|_x^x \tag{9}$$

for all $x \ge 1$. (This nice inequality was rediscovered and proved by different methods in [11].) Using (8) and (9) we get

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) &= \int_1^\infty \frac{1}{n} \sum_{j=1}^{n} s_j(T_n(a))^x \, d\mu(x) \le \int_1^\infty \|a\|_x^x \, d\mu(x) \\ &= \int_1^\infty \int_{-\pi}^{\pi} |a(\theta)|^x \, \frac{d\theta}{2\pi} \, d\mu(x) = \|F(|a|)\|_1 \end{aligned}$$

and hence

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} F(s_j(T_n(a))) \le \|F(|a|)\|_1.$$

Combining this estimate with Proposition 2.1 we arrive at (3). □

Corollary 2.6. *Let I be a finite subset of $[0,1]$ and J be a countable subset of $(1,\infty)$. For $p \in I \cup J$, let F_p be a positive real number. Suppose the series*

$$F(s) := \sum_{p \in I \cup J} F_p s^p$$

converges for every $s \in [0,\infty)$. Then $F : [0,\infty) \to [0,\infty)$ is a continuous and monotonically increasing function and (3) holds for this F and all $a \in L^1$.

Proof. Let $d\mu(x) = \sum_{p \in J} F_p \delta(x-p) \, dx$. Then

$$F(s) = \sum_{p \in I} F_p s^p + \int_1^\infty s^x \, d\mu(x) =: F_1(s) + F_2(s).$$

It is obvious that F_1 is nonnegative, continuous, and monotonically increasing. For F_2, these properties can be deduced from Proposition 2.5. Since $F_1(s) = O(s)$ as $s \to \infty$, result (c) implies that (3) is true for F_1 and all $a \in L^1$. Proposition 2.5 yields (3) for F_2 and all $a \in L^1$. □

Corollary 2.6 shows in particular that (3) is valid for all $a \in L^1$ if

$$F(s) = \exp(\alpha s^\beta) = \sum_{p=0}^{\infty} \frac{\alpha^p}{p!} s^{p\beta}$$

with α and β in $(0,\infty)$. Using Proposition 2.5 with $d\mu(x) = \chi_{(1,\alpha)}(x)\,dx$ $(\alpha > 1)$ we get (3) for all $a \in L^1$ and

$$F(s) = \int_1^\alpha s^x\,dx = \frac{s^\alpha - s}{\log s}.$$

Because $s/\log s = O(s)$ as $s \to \infty$, it follows from (c) that (3) holds for all $a \in L^1$ and $F(s) \simeq \frac{s^\alpha}{\log s}$ $(s \to \infty)$ provided $\alpha \in (1,\infty)$. Analogously (we omit the computational details), the choice $d\mu(x) = \chi_{(1,\alpha)}(x)(\alpha - x)^{\beta-1}\,dx$ $(\alpha > 1, \beta > 0)$ delivers (3) for all $a \in L^1$ and

$$F(s) \simeq \frac{s^\alpha}{(\log s)^\beta} \quad (s \to \infty) \tag{10}$$

with $\alpha \in (1,\infty)$ and $\beta \in (0,\infty)$. Taking $d\mu(x) = \chi_{(1,\alpha)}(x)(\alpha - x)^{\beta-1}|\log(\alpha - x)|^\gamma$ with $\alpha > 1$, $\beta > 0$, $\gamma \in (-\infty,\infty)$, we obtain

$$F(s) \simeq \frac{s^\alpha}{(\log s)^\beta}(\log\log s)^\gamma \quad (s \to \infty), \tag{11}$$

and the measure $d\mu(x) = \chi_{(1,2)}\exp(-1/(2-x))\,dx$ yields

$$F(s) \simeq \frac{s^2}{(\log s)^{3/4}}\exp(-2\sqrt{\log s}) \quad (s \to \infty). \tag{12}$$

Note that these two asymptotic estimates are not trivial but require results from [4]. Thus, if F satisfies (11) or (12) then (3) is true for all $a \in L^1$. Clearly, the restriction to $\alpha > 1$ in (10) and (11) can be dropped since $F(s) = O(s)$ for $\alpha \le 1$.

Finally, if $\mu((\alpha,\beta)) > 0$ for some interval $(\alpha,\beta) \subset (1,\infty)$, then, for $s > 1$,

$$F(s) \ge \int_\alpha^\beta s^x\,d\mu(x) \ge s^\alpha \mu((\alpha,\beta)),$$

which is impossible if $F(s) = O(s(\log s)^\gamma)$ with $\gamma \in \mathbb{R}$ as $s \to \infty$. Consequently, Proposition 2.5 does not give an answer to Problem 1.2.

Remark 2.7. To prove Corollary 2.6 we used that $\int_1^\infty s^x\delta(x-p)dx = s^p$ for $p > 1$. The formula $\int_1^\infty s^x\delta'(x-p)dx = -s^p\log s$ $(p > 1)$ is perhaps a reasonable starting point for an analysis that yields (3) for $F(s) = s^p\log s$ $(p > 1)$ and all $a \in L^1$. Note that again $F(s) = s\log s$ would remain unattained.

3. Determinants of banded Hermitian Toeplitz matrices

While Section 2 was concerned with test functions that do not satisfy the usual growth restrictions at infinity, we now turn to Szegö's formula (1) for $a \in L^\infty$ and $F(\lambda) = \log|\lambda|$. In that case the behavior of $F(\lambda)$ as $|\lambda| \to \infty$ is not of importance. The delicacy comes rather from the singularity of the function at the origin.

For $F(\lambda) = \log|\lambda|$, formula (1) can be written in the form

$$\frac{1}{n}\log|D_n(a)| = \int_{-\pi}^{\pi}\log|a(\theta)|\,\frac{d\theta}{2\pi} + o(1), \tag{13}$$

where $D_n(a) = \det T_n(a)$. If $|a(\theta)| \geq \varepsilon > 0$ for almost all $\theta \in (-\pi, \pi)$, then $|\lambda_j(T_n(a))| \geq \varepsilon > 0$ for all j and n and (13) is known to be true (see, e.g., [5]). We are interested in the question on what happens to (13) if $|a|$ is not bounded away from zero.

Things are fairly transparent for tridiagonal Toeplitz matrices. If $a(\theta) = e^{i\theta} + e^{-i\theta} = 2\cos\theta$, straightforward computation gives

$$D_n(a) = \begin{cases} 0 & \text{if} \quad n \equiv 1, 3 \pmod 4 \\ -1 & \text{if} \quad n \equiv 2 \pmod 4 \\ 1 & \text{if} \quad n \equiv 0 \pmod 4 \end{cases} \qquad \text{and} \qquad \int_{-\pi}^{\pi} \log|a(\theta)| \frac{d\theta}{2\pi} = 0.$$

Thus, (13) is true if (and only if) we agree to define $\log 0 := 0$. Without that agreement, (13) would still hold with the restriction to even n's. The general form of a real-valued trigonometric polynomial of degree 1 is

$$a(\theta) = c + be^{i\theta} + \overline{b}e^{-i\theta} \tag{14}$$

where $c \in \mathbb{R}$ and $b \in \mathbb{C} \setminus \{0\}$. This function can also be written in the form $a(\theta) = c + 2|b|\cos(\theta - \theta_0)$. The eigenvalues of $T_n(a)$ are

$$\lambda_j := \lambda_j(T_n(a)) = c + 2|b|\cos\frac{\pi j}{n+1} \quad (j = 1, \ldots, n) \tag{15}$$

(see [2] or [5]). They densely fill the segment $[c - 2|b|, c + 2|b|]$ as $n \to \infty$. This segment contains the origin, or equivalently, the function a has a real zero, if and only if $|c| \leq 2|b|$, in which case there is a unique $x \in [0, 1]$ such that

$$c + 2|b|\cos\pi x = 0. \tag{16}$$

The degree of approximation of x by rational fractions with n in the denominator is measured by

$$\psi_x(n) := n \min_{j=1,\ldots,n-1} \left| x - \frac{j}{n} \right|.$$

Throughout what follows we define $\log 0 := 0$.

Proposition 3.1. *Let a be given by* (14)*, suppose $|c| \leq 2|b|$, and define $x \in [0, 1]$ by* (16)*. Then*

$$\frac{1}{n}\log|D_n(a)| = \int_0^{2\pi} \log|a(\theta)| \frac{d\theta}{2\pi} + \frac{1}{n}\log\psi_x(n+1) + o(1). \tag{17}$$

Proof. Let $j_n \in \{1, \ldots, n\}$ satisfy $\left| x - \frac{j_n}{n+1} \right| = \frac{\psi_x(n+1)}{n+1}$. From (15) and (16) we infer that, as $n \to \infty$,

$$\begin{aligned} |\lambda_{j_n}| &= 2|b| \left| \cos\frac{\pi j_n}{n+1} - \cos\pi x \right| \sim 2\pi|b|\sin(\pi x) \left| \frac{j_n}{n+1} - x \right| \\ &= 2\pi|b|\sin(\pi x)\, \frac{\psi_x(n+1)}{n+1}, \end{aligned}$$

where $\alpha_n \sim \beta_n$ means that $\alpha_n/\beta_n \to 1$. Hence,

$$\frac{1}{n}\log|\lambda_{j_n}| = \frac{1}{n}\log\frac{\psi_x(n+1)}{n+1} + o(1) = \frac{1}{n}\log\psi_x(n+1) + o(1).$$

Since $|\lambda_j - x| \geq \frac{1}{2(n+1)}$ for $j \neq j_n$, it follows from [9, No. 29 on p. 53] that

$$\begin{aligned} \frac{1}{n}\sum_{j\neq j_n}\log|\lambda_j| &= \frac{1}{\pi}\int_0^\pi \log|c+2|b|\cos\theta|d\theta + o(1) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|c+2|b|\cos\theta|d\theta + o(1), \end{aligned}$$

which completes the proof. □

Thus, (13) holds if and only if $\frac{1}{n}\log\psi_x(n+1) \to 0$ as $n \to \infty$.

Proposition 3.2. *The set $\mathcal{X}$ of all $x \in [0,1]$ for which $\frac{1}{n}\log\psi_x(n+1)$ does not converge to zero as $n \to \infty$ is uncountable, dense, and of measure zero. Rational numbers do not belong to $\mathcal{X}$.*

Proof. We use results that can all be found in [6] and [7]. Clearly, $\frac{1}{n}\log\psi_x(n+1)$ goes to 0 if and only if $\frac{1}{n}\log\psi_x(n) \to 0$. If $x = p/q$ is rational, then $\psi_x(n) = 0$ if n is divisible by q and $\psi_x(n) \geq 1/(qn)$ if n is not divisible by q. Thus, $\frac{1}{n}\log\psi_x(n) \to 0$ and $x \notin \mathcal{X}$. So assume x is irrational.

A function $\varphi : \mathbb{N} \to (0,1]$ is called an approximation function for an irrational number y if there exists a sequence $n_k \to \infty$ such that $\psi_y(n_k) < \varphi(n_k)$. Let $A(\varphi)$ denote the set of all irrational $y \in (0,1)$ for which φ is an approximation function.

Let the irrational number x be in $\mathcal{X}$. Then there exist an $\varepsilon > 0$ and a sequence $n_k \to \infty$ such that $\frac{1}{n_k}|\log\psi_x(n_k)| \geq 2\varepsilon$ for all n_k. Thus, letting $\mathcal{X}_N$ denote the set of all irrational $y \in \mathcal{X}$ for which there is a sequence $n_k \to \infty$ such that $\frac{1}{n_k}|\log\psi_y(n_k)| \geq 2/N$ for all n_k, we have $\mathcal{X} \subset \cup_{N=1}^\infty \mathcal{X}_N$. If $x \in \mathcal{X}_N$, then $|\log\psi_x(n_k)| \geq 2n_k/N$ for some sequence $n_k \to \infty$ and hence we obtain that $\psi_x(n_k) \leq e^{-2n_k/N} < e^{-n_k/N}$. This implies that $x \in A(\varphi_N)$ for $\varphi_N(n) = e^{-n/N}$. Since $\sum_{n=1}^\infty \varphi_N(n) < \infty$, the set $A(\varphi_N)$ has measure zero (Khinchin's theorem). Consequently, $\mathcal{X}_N \subset A(\varphi_N)$ is of measure zero and thus $\mathcal{X}$ also has measure zero.

Finally, take $\varphi(n) = e^{-n^2}$. If $x \in A(\varphi)$, then $\psi_x(n_k) < e^{-n_k^2}$ for some sequence $n_k \to \infty$. For this sequence, $\frac{1}{n_k}|\log\psi_x(n_k)| \to \infty$. Thus, $x \in \mathcal{X}$. We have proved that $A(\varphi) \subset \mathcal{X}$. Since $A(\varphi)$ is known to be uncountable and dense for every function $\varphi : \mathbb{N} \to (0,1]$, it follows that $\mathcal{X}$ is uncountable and dense as well. □

We remark that irrational numbers that are algebraic over $\mathbb{Q}$ do also not belong to $\mathcal{X}$ since, by a theorem of Liouville, $\psi_x(n) \geq cn^{1-\alpha}$ with some $c > 0$ if x is algebraic of degree $\alpha \geq 2$.

The example $a(\theta) = 2\cos\theta$ considered in the beginning motivates the restriction to matrix dimensions n that belong to arithmetic progressions. Here is a result in this direction.

Proposition 3.3. *Let a be given by* (14)*, suppose* $|c| \le 2|b|$*, and define* $x \in [0,1]$ *by* (16)*. For each natural number* $\ell \ge 1$*, there exists a subsequence* $\{n_1, n_2, n_3, \ldots\}$ *of* $\{\ell, 2\ell, 3\ell, \ldots\}$ *such that* $\frac{1}{n_k} \log \psi_x(n_k + 1) \to 0$ *and hence*

$$\frac{1}{n_k} \log |D_{n_k}(a)| = \int_{-\pi}^{\pi} \log |a(\theta)| \frac{d\theta}{2\pi} + o(1) \quad (k \to \infty).$$

Proof. Since $\frac{1}{n} \log \psi_x(n+1) \to 0$ for rational numbers x, we may assume that x is irrational. Let $j_{m\ell+1} \in \{1, \ldots, m\ell\}$ be the number for which $|(m\ell+1)x - j_{m\ell+1}| = \psi_x(m\ell+1)$. We have $\psi_x(m\ell+1) < 1$ and hence $\log \psi_x(m\ell+1) < 0$ for all m. Assume there is no sequence $m_k \to \infty$ such that $\frac{1}{m_k \ell} \log \psi_x(m_k \ell + 1) \to 0$. Then there is an $\varepsilon > 0$ such that $\frac{1}{m\ell} \log \psi_x(m\ell+1) \le -\varepsilon$ and thus $\psi_x(m\ell+1) \le e^{-m\ell\varepsilon}$ for all m. We write x in the base $\ell+1$:

$$x = \frac{x_1}{\ell+1} + \frac{x_2}{(\ell+1)^2} + \frac{x_3}{(\ell+1)^3} + \cdots.$$

Then

$$(\ell+1)^k x = x_1(\ell+1)^{k-1} + \cdots + x_k + \frac{x_{k+1}}{\ell+1} + \frac{x_{k+2}}{(\ell+1)^2} + \cdots$$

and hence

$$j_{(\ell+1)^k} = x_1(\ell+1)^{k-1} + \cdots + x_k \quad \text{or} \quad j_{(\ell+1)^k} = x_1(\ell+1)^{k-1} + \cdots + x_k + 1.$$

If $1 \le x_{k+1} \le \ell - 1$, then

$$\psi_x((\ell+1)^k) = \left|(\ell+1)^k x - j_{(\ell+1)^k}\right| \ge \frac{1}{\ell+1}.$$

Consequently, $\frac{1}{\ell+1} \le e^{-[(\ell+1)^k - 1]\varepsilon}$, which is impossible for $k \ge k_1$. It follows that $x_{k+1} = 0$ or $x_{k+1} = \ell$ for all $k \ge k_1$. Suppose we have $x_{k+1} = 0$ and $x_{k+2} = \ell$. In that case $j_{(\ell+1)^k} = x_1(\ell+1)^k + \cdots + x_k$ and

$$\begin{aligned} \psi_x((\ell+1)^k) &= (\ell+1)^k x_k - j_{(\ell+1)^k} \\ &= \frac{\ell}{(\ell+1)^2} + \frac{x_{k+3}}{(\ell+1)^3} + \frac{x_{k+4}}{(\ell+1)^4} + \cdots \ge \frac{\ell}{(\ell+1)^2}. \end{aligned}$$

As $\frac{\ell}{(\ell+1)^2} \le e^{-[(\ell+1)^k - 1]\varepsilon}$ is not true whenever $k \ge k_2$, we conclude that the combination $x_{k+1} = 0$ and $x_{k+2} = \ell$ is not possible for $k \ge k_2$. Thus, either $x_k = 0$ for all $k \ge k_3$ or $x_k = \ell$ for all $k \ge k_3$. But this is a contradiction to our hypothesis that x be irrational. □

The previous three propositions dealt with trigonometric polynomials of the degree 1. We don't know a useful result on general trigonometric polynomials, but we can say at least the following. For convenience we assume that a is nonconstant and $\|a\|_\infty := \max |a(\theta)| = 1$; the general case can be reduced to this case by multiplying a by an appropriate constant. Our assumption guarantees that all eigenvalues of $T_n(a)$ lie in $(-1, 1)$. We denote by $\|T_n^+(a)\|$ the spectral norm of the Moore-Penrose inverse of $T_n(a)$. Clearly, $\|T_n^+(a)\|$ is nothing but the maximum of the inverses of the absolute values of the nonzero eigenvalues of $T_n(a)$. Thus, $\|T_n^+(a)\| > 1$ for all $n \geq 1$.

Proposition 3.4. *Let a be a real-valued trigonometric polynomial with at least one real zero and with $\|a\|_\infty = 1$. If $\alpha \geq 1$ is the maximal order of the real zeros of a, then*

$$\frac{1}{n}\log|D_n(a)| = \int_{-\pi}^{\pi} \log|a(\theta)|\frac{d\theta}{2\pi} + O\left(\frac{\log\|T_n^+(a)\|}{n^{1/\alpha}}\right) + o(1).$$

Proof. We omit the technical details and confine ourselves to an outline of the basic steps. Let $r \geq 1$ be the degree of the trigonometric polynomial a. We denote by $C_{n+R}(a)$ the circulant matrix of order $n+R$ that can be associated with the banded Toeplitz matrix $T_n(a)$ (see, e.g., [2, p. 33]). Let $\tilde{\lambda}_1 \leq \cdots \leq \tilde{\lambda}_{n+R}$ be the eigenvalues of $C_{n+R}(a)$ and denote by $\lambda_1 \leq \cdots \leq \lambda_n$ the eigenvalues of $T_n(a)$. By Cauchy's interlacing theorem, $\tilde{\lambda}_j \leq \lambda_j \leq \tilde{\lambda}_{j+R}$ for $j = 1, \ldots, n$. The eigenvalues of $C_{n+R}(a)$ are

$$a\left(\frac{2\pi k}{n+R}\right) \quad (k = 0, \ldots, n+R-1).$$

Since $|a'(\theta)|$ is bounded, we have

$$\left|a\left(\frac{2\pi(k+1)}{n+R}\right) - a\left(\frac{2\pi k}{n+R}\right)\right| = |a'(\theta_k)|\,\frac{2\pi}{n+R} \leq \frac{M}{n+R}$$

with some constant $M < \infty$ for all k. We can show that

$$\#\left\{j : |\lambda_j| < \frac{3M}{n+R}\right\} = O(n^{1-1/\alpha}), \tag{18}$$

where $\#E$ denotes the number of elements of a finite set E. Let $F(\lambda) = \log|\lambda|$. Using (18) we get

$$\left|\sum_{|\lambda_j|<\frac{3M}{n+R}} \frac{F(\lambda_j)}{n}\right| = O\left(\frac{\log\|T_n^+(a)\|}{n^{1/\alpha}}\right).$$

Furthermore, we can prove that

$$\sum_{\lambda_j \leq -\frac{3M}{n+R}} \frac{|F(\lambda_j) - F(\tilde{\lambda}_j)|}{n} + \sum_{\lambda_j \geq \frac{3M}{n+R}} \frac{|F(\lambda_j) - F(\tilde{\lambda}_{j+R})|}{n} = o(1).$$

Making use of [9, No. 29 on p. 53] we finally obtain that

$$\sum_{\lambda_j \leq -\frac{3M}{n+R}} \frac{F(\tilde{\lambda}_j)}{n} + \sum_{\lambda_j \geq \frac{3M}{n+R}} \frac{F(\tilde{\lambda}_{j+R})}{n} = \int_{-\pi}^{\pi} \log|a(\theta)|\,\frac{d\theta}{2\pi} + o(1).$$

Putting the things together we arrive at the asserted formula. □

Proposition 3.4 tells us that (13) is certainly true if $\|T_n^+(a)\|$ increases at most polynomially. If, in particular, $a(\theta) \geq 0$ for all θ or $a(\theta) \leq 0$ for all θ, then $\|T_n^+(a)\| = \|T_n^{-1}(a)\| = O(n^\alpha)$ (see [2, Corollary 4.34]) and hence (13) is valid.

The following problem (which is Problem 1.1 in the case $\ell = 1$) is perhaps the simplest one should tackle to gain more insight into the matter.

Problem 3.5. *Let a be a real-valued trigonometric polynomial that assumes both positive and negative values and let $\ell \geq 1$ be a natural number. Is there a subsequence $\{n_1, n_2, n_3, \ldots\}$ of $\{\ell, 2\ell, 3\ell, \ldots\}$ such that*

$$\frac{1}{n_k} \log |D_{n_k}(a)| = \int_{-\pi}^{\pi} \log |a(\theta)| \frac{d\theta}{2\pi} + o(1) \quad (k \to \infty) \ ?$$

4. Higher dimensions and block case

The problems considered so far are of even greater interest for higher-dimensional block Toeplitz operators. Let $a : (-\pi, \pi)^d \to \mathbb{C}^{N \times N}$ be in L^1 on $(-\pi, \pi)^d$ and put

$$a_j = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} a(\theta_1, \ldots, \theta_d) e^{-i(j_1\theta_1 + \cdots + i_d\theta_d)} \, d\theta_1 \ldots d\theta_d$$

for $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$. We denote by $T_n(a)$ the operator acting on the ℓ^2 space of all functions $\varphi : \{1, \ldots, n\}^d \to \mathbb{C}^N$ by the rule

$$(T_n(a)\varphi)_j = \sum_{k \in \{1,\ldots,n\}^d} a_{j-k} \varphi_k, \quad j \in \{1, \ldots, n\}^d.$$

After preliminary work by many authors, Tilli [12] found textbook proofs of the following formulas: if $a = a^*$ then

$$\lim_{n \to \infty} \frac{1}{n^d} \sum_{j=1}^{Nn^d} F(\lambda_j(T_n(a))) = \int_{(-\pi,\pi)^d} \sum_{k=1}^{N} F(\lambda_k(a(\theta))) \frac{d\theta}{(2\pi)^d}$$

and for general a we have

$$\lim_{n \to \infty} \frac{1}{n^d} \sum_{j=1}^{Nn^d} F(s_j(T_n(a))) = \int_{(-\pi,\pi)^d} \sum_{k=1}^{N} F(s_k(a(\theta))) \frac{d\theta}{(2\pi)^d}$$

provided that $F : \mathbb{R} \to \mathbb{R}$ is bounded and uniformly continuous.

It follows in particular that if $a : (-\pi, \pi)^d \to \mathbb{C}^{N \times N}$ is a matrix-valued trigonometric polynomial in d variables such that $a(\theta)$ is a positive definite Hermitian matrix for every $\theta \in (-\pi, \pi)^d$, then

$$\frac{1}{n^d} \log |D_n(a)| = \int_{(-\pi,\pi)^d} \log |\det a(\theta)| \frac{d\theta}{(2\pi)^d} + o(1). \tag{19}$$

Now suppose

$$a(\theta_1, \theta_2) = w + ue^{i\theta_1} + u^* e^{-i\theta_1} + ve^{i\theta_2} + v^* e^{-i\theta_2} \tag{20}$$

where $v \in \mathbb{C}^{N \times N}$ is Hermitian and u, v are arbitrary matrices in $\mathbb{C}^{N \times N}$. Then $a = a^*$. From Section 3 we know that we cannot expect (19) to be true for indefinite matrix functions. Here is a precise result in this direction.

Proposition 4.1. *Let $\ell \geq 1$ be a natural number. There exist scalar-valued functions of the form* (20) *and a subsequence $\{n_1, n_2, n_3, \ldots\}$ of $\{\ell, 2\ell, 3\ell, \ldots\}$ such that* $\log|a|$ *is in L^1 but*

$$\lim_{k\to\infty} \frac{1}{n_k^2} \log|D_{n_k}(a)| = \infty.$$

Proof. Let

$$a(\theta_1,\theta_2) = a_1(\theta_1)a_2(\theta_2) = (c_1 + b_1 e^{i\theta_1} + \overline{b}_1 e^{-i\theta_1})(c_2 + b_2 e^{i\theta_2} + \overline{b}_2 e^{-i\theta_2})$$

with $c_j \in \mathbb{R}$ and $b_j \in \mathbb{C}$. Clearly, $\log|a| \in L^1$. Suppose $|c_j| \leq 2|b_j|$ and define $x_j \in [0,1]$ by $c_j + 2|b_j|\cos \pi x_j = 0$. We have $T_n(a) = T_n(a_1) \otimes T_n(a_2)$ and hence $D_n(a) = D_n(a_1)^n D_n(a_2)^n$. Thus,

$$\frac{1}{n^2}\log|D_n(a)| = \frac{1}{n}\log|D_n(a_1)| + \frac{1}{n}\log|D_n(a_2)|.$$

Taking into account Proposition 2.1 and the identity

$$\int_{-\pi}^{\pi} \log|a_1(\theta_1)| \frac{d\theta_1}{2\pi} + \int_{-\pi}^{\pi} \log|a_2(\theta_2)| \frac{d\theta_2}{2\pi} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log|a(\theta_1,\theta_2)| \frac{d\theta_1 d\theta_2}{(2\pi)^2}$$

we therefore see that $\frac{1}{n^2}\log|D_n(a)|$ equals

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log|a(\theta_1,\theta_2)| \frac{d\theta_1 d\theta_2}{(2\pi)^2} + \frac{1}{n}\log\psi_{x_1}(n+1) + \frac{1}{n}\log\psi_{x_2}(n+1) + o(1).$$

We are so left with showing that there are $x \in [0,1]$ and $m_1 < m_2 < \cdots$ such that $\frac{1}{m_k}\log\psi_x(\ell m_k + 1) \to \infty$. But if x is a number whose representation in the base $\ell + 1$ is of the form $x = 0.10\ldots010\ldots010\ldots010\ldots$ with sufficiently long chains of zeros, then $\frac{1}{m_k}\log\psi_x(\ell m_k + 1) \to \infty$ if the numbers $\ell m_k + 1$ are chosen as appropriate powers of $\ell + 1$. □

We conclude with two open questions. The first of them can perhaps be tackled as in the proof of Proposition 3.4 by comparing Toeplitz matrices with appropriate circulants. The second of these questions seems to be harder and is in fact the thing one really wants to know.

Problem 4.2. *Let $a : (-\pi,\pi]^d \to \mathbb{C}^{N\times N}$ be a trigonometric matrix polynomial such that $a(\theta)$ is a Hermitian matrix with all eigenvalues in $(-1,1)$ for every $\theta \in (-\pi,\pi]^d$. Suppose also that* $\det a$ *has at least one zero in $(-\pi,\pi]^d$. Is there a number $\beta \in (0,\infty)$ such that*

$$\frac{1}{n^d}\log|D_n(a)| = \frac{1}{2\pi}\int_{(-\pi,\pi)^d} \log|\det a(\theta)| \frac{d\theta}{(2\pi)^d} + O\left(\frac{\log\|T_n^+(a)\|}{n^\beta}\right) + o(1) \ ?$$

Problem 4.3. *Let a be of the form* (20) *and let $\ell \geq 1$ be a natural number. Is there a subsequence $\{n_1, n_2, n_3, \ldots\}$ of $\{\ell, 2\ell, 3\ell, \ldots\}$ such that $D_{n_k}(a) \neq 0$ for all k and*

$$\frac{1}{n_k^2}\log|D_{n_k}(a)| = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log|\det a(\theta_1,\theta_2)| \frac{d\theta_1 d\theta_2}{(2\pi)^2} + o(1)$$

as $k \to \infty$?

Added in proof. Problems 1.2 and 2.4 were recently solved in [3]. The answer to Problem 1.2 is in the affirmative, while that to Problem 2.4 is negative.

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On the Numerical Solution of a Hypersingular Integral Equation with Fixed Singularities

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Abstract. For the numerical solution of the hypersingular integral equation of a notched half-plane problem we propose collocation methods which look for an approximation of the derivative of the solution of the original equation. This derivative is the solution of a Cauchy singular integral equation with additional fixed singularities. We also give a solvability analysis of the original equation which motivates the suggested numerical methods.

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Keywords. Singular integral equations, fixed singularities, collocation.

1. Introduction

Consider the hypersingular integral equation

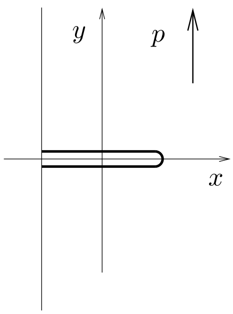
$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{(y-x)^2}+h(x,y)\right]u(y)\,dy=f(x)\,,\quad -1<x<1\,, \tag{1.1}$$

where the finite part integral has to be understood in the sense of Hadamard as the derivative of a Cauchy principal value integral,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u(y)}{(y-x)^2}\,dy=\frac{d}{dx}\frac{1}{\pi}\int_{-1}^{1}\frac{u(y)}{y-x}\,dy\,. \tag{1.2}$$

Moreover,

$$h(x,y)=-\frac{1}{(2+y+x)^2}+\frac{12(1+x)}{(2+y+x)^3}-\frac{12(1+x)^2}{(2+y+x)^4}\,.$$



This equation is obtained by A.C. Kaya and F. Erdogan [11, (57), (72)] considering a notched elastic half-plane: A straight crack of normalized length 2 ends at the boundary of an elastic half-plane at right angle, the faces of the cut and the edge of the half-plane are free from external forces and the medium is subjected at infinity to tensile forces p perpendicular to the line of the cut. The region occupied by the elastic medium is placed in the half-plane $\left\{(x,y) \in \mathbb{R}^2 : x \geq -1\right\}$ (see the figure). The right-hand side in (1.1) then equals $f(x) = -\dfrac{1+\kappa}{2\mu}\, p(x)$, where the elastic constants μ and κ are the shearing modulus and Muskhelishvili's constant, respectively. The unknown function $u(x)$ is the crack opening displacement. Hence, we are interested in bounded solutions $u(x)$ of equation (1.1) vanishing at the point 1,

$$u(1) = 0\,. \tag{1.3}$$

The function $h(x,y)$ can be written in the form

$$h(x,y) = -\frac{1}{(2+y+x)^2} + \frac{12(1+y)}{(2+y+x)^3} - \frac{12(1+y)^2}{(2+y+x)^4} = \frac{\partial}{\partial x} h_0(x,y)$$

with

$$\begin{aligned} h_0(x,y) &= \frac{1}{2+y+x} - \frac{6(1+y)}{(2+y+x)^2} + \frac{4(1+y)^2}{(2+y+x)^3} \\ &= -\frac{1}{2+y+x} - \frac{2(1+x)}{(2+y+x)^2} + \frac{4(1+x)^2}{(2+y+x)^3} \\ &= -\frac{1}{2+y+x} - \frac{2(y-x)(1+x)}{(2+y+x)^3}\,. \end{aligned}$$

Consequently, using (1.2) and applying the integration operator $\displaystyle\int_0^{x_0} dx$ to equation (1.1), this equation can be written in the form

$$\left(\widetilde{A}u\right)(x) := \frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x} + h_0(x,y) - h_1(y)\right] u(y)\,dy = F(x)\,, \quad -1 < x < 1\,, \tag{1.4}$$

where $F(x) = \displaystyle\int_0^x f(y)\,dy$ and

$$h_1(y) = \frac{1}{y} + h_0(0,y) = \frac{1}{y} - \frac{1}{2+y} - \frac{2y}{(2+y)^3}\,.$$

The operator $\widehat{A}$ defined by

$$\left(\widehat{A}u\right)(x) := \frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}+h_0(x,y)\right]u(y)\,dy\,,\quad -1<x<1\,, \tag{1.5}$$

(cf. (1.4)) is the integral operator of the equation

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}+h_0(x,y)\right]u(y)\,dy = g(x)\,,\quad -1<x<1\,, \tag{1.6}$$

of a notched half-plane considered in [10, (37a.10), (37a.10′)], where the right-hand side is of the form

$$g(x) = g_1(1+x)+g_0 \tag{1.7}$$

with a given constant g_1, but an unknown constant g_0. If we assume that the solution $u(x)$ of (1.1) has a p-summable derivative $u'(x)$ for some $p>1$, then, due to (1.3) and due to [14, Chapter II, Lemma 6.1],

$$\int_{-1}^{1}\frac{u(y)}{(y-x)^2}\,dy = -\frac{u(-1)}{1+x}+\int_{-1}^{1}\frac{u'(y)}{y-x}\,dy\,,\quad -1<x<1\,.$$

Furthermore, integration by part, again using (1.3), gives

$$\int_{-1}^{1}\frac{u(y)}{(2+y+x)^{k+1}}\,dy = \frac{u(-1)}{k(1+x)^k}+\frac{1}{k}\int_{-1}^{1}\frac{u'(y)}{(2+y+x)^k}\,dy\,,$$

$x>-1\,,\ k=1,2,\ldots$. Consequently, $v(x)=u'(x)$ is a solution of the equation

$$(Bv)(x) = f(x) \tag{1.8}$$

where

$$(Bv)(x) = \frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}-\frac{1}{2+y+x}+\frac{6(1+x)}{(2+y+x)^2}-\frac{4(1+x)^2}{(2+y+x)^3}\right]v(y)\,dy\,.$$

This equation can also be written in the form

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}+\frac{1}{2+y+x}+\frac{2(1+y)}{(2+y+x)^2}-\frac{4(1+y)^2}{(2+y+x)^3}\right]v(y)\,dy = f(x)$$

or as

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x}+\frac{1}{2+y+x}-\frac{2(1+y)}{(2+y+x)^2}+\frac{4(1+x)(1+y)}{(2+y+x)^3}\right]v(y)\,dy = f(x)$$

(cf. [2, equ. (14.3)]). For practical purposes the so called stress intensity factor at point $x=1$ defined by

$$k_1(1) = \frac{2\mu}{\kappa+1}\lim_{x\to 1-0}\frac{u(x)}{\sqrt{2(1-x)}}$$

is of main interest in fracture mechanics (cf. [11, (75)]). In terms of the solution $v(x)=u'(x)$ of equation (1.8) we get, taking into account (1.3),

$$k_1(1) = -\frac{2\mu}{\kappa+1}\lim_{x\to 1-0}\left[u'(x)\sqrt{2(1-x)}\right]\,. \tag{1.9}$$

The remaining part of the paper is organized as follows. In Section 2 we discuss some solvability properties of the operator equations presented here in weighted spaces of summable functions. Basing on these results, in Section 3 we propose collocation methods for the approximate solution of equation (1.8). Section 4 is devoted to the effective realization of the collocation methods utilizing special structural properties of the matrices of the respective discretized equations. Finally, in Section 5 we present numerical results paying attention not only to the computation of the stress intensity factor but also to the condition of the discrete systems, since these condition numbers are important for the application of a Krylov subspace iteration method, the CGNR-algorithm, to solve the systems.

2. Solvability properties

Let $v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ be a Jacobi weight, $\alpha,\beta \in (-1,p-1)$. For $1 < p < \infty$, by $\mathbf{L}^p_{\alpha,\beta} = \mathbf{L}^p(v^{\alpha,\beta})$ we denote the Banach space of all (w.r.t. the weight $v^{\alpha,\beta}(x)$) p-summable functions $u : (-1,1) \longrightarrow \mathbb{C}$ equipped with the norm

$$\|u\|_{\mathbf{L}^p_{\alpha,\beta}} = \left(\int_{-1}^1 |u(t)|^p v^{\alpha,\beta}(x)\, dx \right)^{\frac{1}{p}} .$$

Moreover, let $\mathbf{L}^p = \mathbf{L}^p(v^{0,0})$ and $\mathbf{L}^0_1 = \bigcup\limits_{1<p<\infty} \mathbf{L}^p$. We also use the notations $\mathbf{L}^p_\omega = \mathbf{L}^p(\omega)$ and

$$\varphi(x) = \sqrt{1-x^2}\,, \quad \sigma(x) = \frac{1}{\sqrt{1-x^2}}\,, \quad \mu(x) = \sqrt{\frac{1-x}{1+x}}\,, \quad \nu(x) = \sqrt{\frac{1+x}{1-x}}\,.$$

In case $p=2$, the space $\mathbf{L}^2_\omega$ is equipped with the inner product

$$\langle u, v\rangle_\omega = \int_{-1}^1 u(x)\overline{v(x)}\omega(x)\, dx$$

and the associated norm $\|u\|_\omega = \sqrt{\langle u,u\rangle_\omega}$. In [2, Theorem 14.1] we find the following solvability properties of equations (1.6) and (1.8).

Proposition 2.1. *Let $\widehat{A}$ be the operator defined in* (1.5). *Then*

(a) $\widehat{A} : \mathbf{L}^0_1 \longrightarrow \mathbf{L}^0_1$ *is a Fredholm operator with index* 1, *where* $\ker(\widehat{A}) = \operatorname{span}\{u_0\}$ *with* $v^{\frac{1}{2},0}u_0 \in \mathbf{C}^\infty(-1,1) \cap \mathbf{H}_{\frac{1}{2}}(-1,1]$, *and* $u_0(t)$ *is bounded in* $(-1,0)$. *By* $\mathbf{H}_\rho$ *there are denoted Hölder classes, i.e., for example,*

$$\mathbf{H}_\rho(a,b] = \left\{ f : (a,b] \longrightarrow \mathbb{C} \text{ with } \sup_{a<x<y\le b} \left\{ \frac{|f(x)-f(y)|}{|x-y|^\rho} \right\} < \infty \right\} .$$

$\mathbf{C}^\infty(a,b)$ *denotes the set of all continuous functions* $f : (a,b) \longrightarrow \mathbb{C}$ *for which all derivatives* $f^{(k)} : (a,b) \longrightarrow \mathbb{C}$, $k = 1,2,\dots$, *exist.*

(b) *Let* $2 < p < \infty$. *For each* $f \in \mathbf{L}^p(v^{0,\beta})$, *equation* (1.6) *has a unique solution* $u \in \mathbf{L}^p(v^{0,\beta})$. *If* $g(x)$ *is of the form* (1.7) *then* $u(1) = 0$, $u \in \mathbf{C}^\infty(-1,1)$, *and* $u : (-1,1] \longrightarrow \mathbb{C}$ *is bounded.*

(c) *In case* $f(x) \equiv 1$ *the solution* $v(x)$ *of equation* (1.8) *is bounded at the point* $x = -1$ *and* $v^{\frac{1}{2},0}v \in \mathbf{C}^\infty(-1,1) \cap \mathbf{H}_{\frac{1}{2}}(-1,1]$.

We are also interested in solvability properties of equations (1.4), (1.6), and (1.8) in $\mathbf{L}^2(v^{\alpha,\beta})$ spaces with $\alpha \neq 0$. For this define $\gamma = \dfrac{1+\alpha}{2}$ and $\delta = \dfrac{1+\beta}{2}$,

$$a_\gamma(\xi) = -\coth \pi(\mathrm{i}\gamma + \xi), \quad -\infty < \xi < \infty,$$

and

$$b_\delta^{\widehat{A}}(\xi) = \coth \pi(\mathrm{i}\delta + \xi) + n_{0,\delta}(\xi) - 6\, n_{1,\delta}(\xi) + 4\, n_{2,\delta}(\xi), \quad -\infty < \xi < \infty,$$

and

$$b_\delta^{B}(\xi) = \coth \pi(\mathrm{i}\delta + \xi) + n_{0,\delta}(\xi) + 2\, n_{1,\delta}(\xi) - 4\, n_{2,\delta}(\xi), \quad -\infty < \xi < \infty,$$

where

$$n_{k,\delta}(\xi) = (-1)^k \binom{\delta - \mathrm{i}\xi - 1}{k} \frac{1}{\sinh \pi(\mathrm{i}\delta + \xi)}.$$

Consider the oriented (closed) curves

$$\Gamma_{\gamma,\delta}^{\widehat{A}} = \{a_\gamma(\xi) : -\infty \le \xi < \infty\} \cup \left\{b_\delta^{\widehat{A}}(\xi) : -\infty \le \xi < \infty\right\}$$

and

$$\Gamma_{\gamma,\delta}^{B} = \{a_\gamma(\xi) : -\infty \le \xi < \infty\} \cup \left\{b_\delta^{B}(\xi) : -\infty \le \xi < \infty\right\}.$$

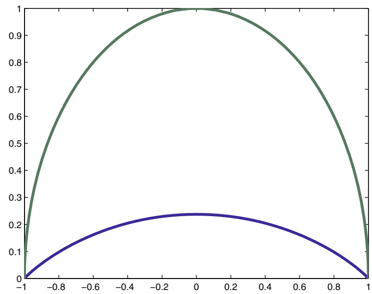
If $0 \notin \Gamma$, by $\operatorname{wind}\Gamma$ we denote the winding number of the closed curve Γ w.r.t. to the origin.

Proposition 2.2 ([2], Theorem 9.1). *The operator* $\widehat{A} : \mathbf{L}^2(v^{\alpha,\beta}) \longrightarrow \mathbf{L}^2(v^{\alpha,\beta})$ *is Fredholm if and only if* $0 \notin \Gamma_{\gamma,\delta}^{\widehat{A}}$, *where* $\operatorname{ind}\widehat{A} = -\operatorname{wind}\Gamma_{\gamma,\delta}^{\widehat{A}}$. *Moreover, if the operator* $\widehat{A} : \mathbf{L}^2(v^{\alpha,\beta}) \longrightarrow \mathbf{L}^2(v^{\alpha,\beta})$ *is Fredholm then it is at least one-sided invertible. The same holds true for the operator* $B : \mathbf{L}^2(v^{\alpha,\beta}) \longrightarrow \mathbf{L}^2(v^{\alpha,\beta})$ *if* $\Gamma_{\gamma,\delta}^{B}$ *is used instead of* $\Gamma_{\gamma,\delta}^{\widehat{A}}$.

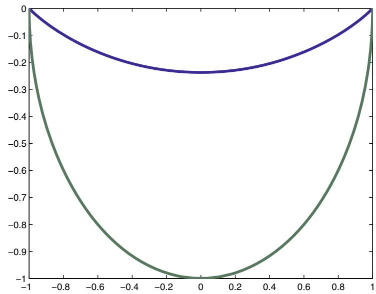
From the investigations of [2, Section 14] we get, for $\gamma \neq \dfrac{1}{2}$,

$$\operatorname{wind}\Gamma_{\gamma,\delta}^{\widehat{A}} = \frac{\operatorname{sgn}(\gamma^{-1} - 2) - 1}{2} \quad \text{and} \quad \operatorname{wind}\Gamma_{\gamma,\delta}^{B} = \frac{1 + \operatorname{sgn}(\gamma^{-1} - 2)}{2}. \tag{2.1}$$

The following figures show $\Gamma_{\gamma,\delta}^{\widehat{A}}$ in case $\alpha = \beta = -\frac{1}{2}$ and $\Gamma_{\gamma,\delta}^{B}$ in case $\alpha = \beta = \frac{1}{2}$.



$\Gamma^{\widehat{A}}_{\gamma,\delta}$ in case $\alpha = -\frac{1}{2}$, $\beta = -\frac{1}{2}$



$\Gamma^{B}_{\gamma,\delta}$ in case $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$

Corollary 2.3. *The operator* $B : \mathbf{L}^2_\varphi \longrightarrow \mathbf{L}^2_\varphi$ *is invertible, i.e., equation* (1.8) *has for each right-hand side* $f \in \mathbf{L}^2_\varphi$ *a unique solution* $v \in \mathbf{L}^2_\varphi$. *The same is true for the operator* $\widehat{A} : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ *of the equation* (1.6).

Remark 2.4. *For each solution* $v \in \mathbf{L}^2_\varphi$ *of* (1.8) *the function* $u(x) = -\int_x^1 v(y)\,dy$ *is a solution of* (1.4) *as well as of* (1.1). *This follows from the fact, that* $v \in \mathbf{L}^2_\varphi$ *implies* $v \in \mathbf{L}^p$ *for* $1 \le p < \frac{3}{4}$, *and from the considerations in Section* 1.

By $\mathbf{L}^{2,1}_\sigma$ we denote the space $\left\{f \in \mathbf{L}^2_\sigma : f' \in \mathbf{L}^2_\varphi\right\}$ equipped with the norm

$$\|f\|_{\mathbf{L}^{2,1}_\sigma} = \sqrt{\|f\|^2_\sigma + \|f'\|^2_\varphi}\,.$$

Since $f' \in \mathbf{L}^2_\varphi$ is equivalent to $f'\varphi^2 \in \mathbf{L}^2_\sigma$ the Sobolev like space $\mathbf{L}^{2,1}_\sigma$ belongs to a scale of Hilbert spaces introduced, for example, in [1] (cf. [1, pp. 196, 197]). For $t \in [-1,1]$, let $\mathbf{L}^{2,1}_{\sigma,t}$ be the subspace of such functions $u \in \mathbf{L}^{2,1}_\sigma$ satisfying $u(t) = 0$.

Proposition 2.5. *The operator* $\widetilde{A} : \mathbf{L}^{2,1}_{\sigma,1} \longrightarrow \mathbf{L}^{2,1}_{\sigma,0}$ *defined in* (1.4) *is bounded and invertible.*

Proof. By the considerations in Section 1 and by Remark 2.4 we have, for $u \in \mathbf{L}^{2,1}_{\sigma,1}$,

$$(\widetilde{A}u)(x) = \int_0^x (Bu')(y)\,dy$$

such that, due to $B \in \mathcal{L}(\mathbf{L}^2_\varphi)$, $|(\widetilde{A}u)(x)| \le \text{const}\,\|u'\|_{\mathbf{L}^2_\varphi}$. Consequently,

$$\|\widetilde{A}u\|_{\mathbf{L}^{2,1}_\sigma} = \sqrt{\|\widetilde{A}u\|^2_\sigma + \|Bu'\|^2_\varphi} \le \text{const}\,\|u\|_{\mathbf{L}^{2,1}_\sigma} \quad \forall\, u \in \mathbf{L}^{2,1}_{\sigma,1}\,.$$

It remains to show that $\widetilde{A} : \mathbf{L}^{2,1}_{\sigma,1} \longrightarrow \mathbf{L}^{2,1}_{\sigma,0}$ is bijective. If $u \in \mathbf{L}^{2,1}_{\sigma,1}$ and $\widetilde{A}u \equiv 0$ then $Bu' \equiv 0$ and, in view of Corollary 2.3, $u' \equiv 0$. This implies $u \equiv 0$. Now, let $f \in \mathbf{L}^{2,1}_{\sigma,0}$. Then, again due to Corollary 2.3, there exists a $v \in \mathbf{L}^2_\varphi$ with $Bv = f'$, which implies $f(x) = \int_0^x (Bv)(y)\,dy$, $x \in [-1,1]$, and $u(x) = -\int_x^1 v(y)\,dy$ is a solution of $\widetilde{A}u = f$. □

In view of Proposition 2.1,(c) we expect that the solution of (1.8) is of the form

$$v(x) = \frac{v_0(x)}{\sqrt{1-x}} \tag{2.2}$$

with a bounded function $v_0 : [-1,1] \longrightarrow \mathbb{R}$. The operator $J : \mathbf{L}^2_\varphi \longrightarrow \mathbf{L}^2_\mu$, $f \mapsto v^{0,\frac{1}{2}} f$ is an isometrical isomorphism. Consequently, equation (1.8) considered in $\mathbf{L}^2_\varphi$ is equivalent to the equation $JBJ^{-1}u = Jf =: \widetilde{f}$ considered in $\mathbf{L}^2_\mu$, where $u = Jv$. If we write the kernel function of the integral operator B in the form

$$\frac{1}{y-x} + h_B\left(\frac{1+x}{1+y}\right)\frac{1}{1+y} \quad \text{with} \quad h_B(x) = -\frac{1}{1+x} + \frac{6x}{(1+x)^2} - \frac{4x^2}{(1+x)^3}$$

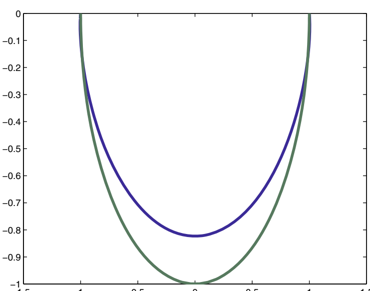
then the kernel function of the integral operator $A := JBJ^{-1}$ is equal to

$$\begin{aligned}
&\sqrt{\frac{1+x}{1+y}}\left[\frac{1}{y-x} + h_B\left(\frac{1+x}{1+y}\right)\frac{1}{1+y}\right] \\
&\quad = \frac{1}{y-x} - \frac{1}{y-x}\left(1 - \sqrt{\frac{1+x}{1+y}}\right) + \sqrt{\frac{1+x}{1+y}}\, h_B\left(\frac{1+x}{1+y}\right)\frac{1}{1+y} \qquad (2.3) \\
&\quad = \frac{1}{y-x} + h_A\left(\frac{1+x}{1+y}\right)\frac{1}{1+y} \quad \text{with} \quad h_A(x) = \sqrt{x}\, h_B(x) - \frac{1}{1+\sqrt{x}}\,.
\end{aligned}$$

As a consequence of these considerations, in the remaining part of this paper we propose and investigate collocation methods for the numerical solution of the operator equation

$$(Au)(x) = \frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{y-x} + h_A\left(\frac{1+x}{1+y}\right)\frac{1}{1+y}\right]u(y)\,dy = \widetilde{f}(x)\,, \tag{2.4}$$

$-1 < x < 1$, in the Hilbert space $\mathbf{L}^2_\mu$. Moreover, since equation (1.8) is also uniquely solvable in $\mathbf{L}^2_\mu$ (cf. Prop. 2.2 and the following figure) we will try to apply the same methods and compared to it two further collocation methods directly to (1.8), too.



$\Gamma^B_{\gamma,\delta}$ in case $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$

Remark 2.6. *Kalandiya* [10, Sect. 37a] *considers the notched half-plane problem in the form* (1.6), (1.7) *with the unknown constant* g_0 *and proposes a collocation-quadrature method for the numerical solution of* (1.6) *respectively for the equation transformed with the help of the operator* J. *But, the linear functional*

$$g_0(u) = \frac{1}{\pi}\int_{-1}^{1} h_1(y)u(y)\,dy$$

is not bounded in appropriate weighted $\mathbf{L}^2$ *spaces. Moreover, Kalandiya took not into account that the integral operator with the kernel function* $h_0(x,y)$ *is not compact in such spaces. Indeed, due to Prop.* 2.2 *the integral equation* (1.6) *is uniquely solvable in* $\mathbf{L}^2_{\alpha,\beta}$ *for each right-hand side* $g \in \mathbf{L}^2_{\alpha,\beta}$ *and* $-1 < \alpha < 0$, $-1 < \beta < 1$. *For* $0 < \alpha < 1$, $-1 < \beta < 1$, *equation* (1.6) *is solvable for each right-hand side* $g \in \mathbf{L}^2_{\alpha,\beta}$ *and the respective homogeneous equation has a one-dimensional solution space in* $\mathbf{L}^2_{\alpha,\beta}$. *Hence, there is no free condition for the determination of the unknown constant* g_0 *in* (1.7) (*cf. also Prop.* 2.1, (a), (b)).

3. Collocation methods

We look for an approximate solution of an integral equation of the form

$$(Au)(x) := a(x)u(x) + \frac{b(x)}{\pi}\int_{-1}^{1}\frac{u(y)}{y-x}\,dy + \frac{1}{\pi}\int_{-1}^{1} h\left(\frac{1+x}{1+y}\right)\frac{dy}{1+y} = f(x)\,,$$

$-1 < x < 1$, with piecewise continuous functions $a, b : [-1,1] \longrightarrow \mathbb{C}$ and a continuous function $h : (0,\infty) \longrightarrow \mathbb{C}$ as

$$u_n(x) = \sum_{k=1}^{n} \xi_{nk}\widetilde{\ell}^{\tau}_{nk}(x)\,, \tag{3.1}$$

where

$$\widetilde{\ell}^{\tau}_{nk}(x) = \frac{\vartheta(x)}{\vartheta(x^{\tau}_{nk})}\,\ell^{\tau}_{nk}(x)\,, \quad \ell^{\tau}_{nk}(x) = \frac{\widehat{p}^{\tau}_{n}(x)}{(x - x^{\tau}_{nk})\,(\widehat{p}^{\tau}_{n})'\,(x^{\tau}_{nk})}\,.$$

Here, for $\tau = \sigma$ or $\tau = \varphi$, $\widehat{p}^{\tau}_{n}(x)$ and x^{τ}_{nk} denote the nth Chebyshev polynomial of first or second kind and the respective Chebyshev nodes of first or second kind

$$\widehat{T}_n(\cos s) = \cos(ns)\,,\ n = 0,1,2,\dots\,, \quad x^{\sigma}_{nk} = \cos\frac{(2k-1)\pi}{2\,n}\,,\ k = 1,\dots,n\,.$$

and

$$\widehat{U}_n(\cos s) = \frac{\sin(n+1)s}{\sin s}\,, \quad n = 0,1,2,\dots\,, \quad x^{\varphi}_{nk} = \cos\frac{k\,\pi}{n+1}\,,\ k = 1,\dots,n\,.$$

The respective normalized (w.r.t. the inner product $\langle .,. \rangle_{\tau}$) Chebyshev polynomials $p^{\tau}_{n}(x)$ of first and second kind are

$$T_0(x) = \frac{1}{\sqrt{\pi}}\,, \quad T_n(\cos s) = \sqrt{\frac{2}{\pi}}\,\cos(n\,s)\,, \quad n = 1,2,\dots$$

and

$$U_n(\cos s) = \sqrt{\frac{2}{\pi}}\, \frac{\sin(n+1)s}{\sin s}\,, \quad n = 0, 1, 2, \ldots$$

Having in mind the representation (2.2) for a solution $v(x)$ of equation (1.8) and $u = Jv$ for a solution $u(x)$ of equation (2.4), for $\vartheta(x)$ we choose

$$\vartheta(x) = \nu(x) = \sqrt{\frac{1+x}{1-x}} = v^{-\frac{1}{2},\frac{1}{2}}(x) \quad \text{or} \quad \vartheta(x) = \frac{1}{\sqrt{1-x}} = v^{-\frac{1}{2},0}(x)\,.$$

With these notations the collocation methods under consideration can be written as

$$a(x^\tau_{nj})\xi_{nj} + \frac{1}{\pi}\int_{-1}^{1}\left[\frac{b(x^\tau_{nj})}{y - x^\tau_{nj}} + h\left(\frac{1+x^\tau_{nj}}{1+y}\right)\frac{1}{1+y}\right] u_n(y)\, dy = f(x^\tau_{nj})\,,$$

$j = 1, \ldots, n$, or shortly as

$$A_n u_n = f_n =: M^\tau_n f\,, \quad A_n = M^\tau_n A|_{\operatorname{im} L_n}\,, \tag{3.2}$$

where

$$M^\tau_n f = \sum_{k=1}^{n} f(x^\tau_{nk})\widetilde{\ell}^\tau_{nk} = \vartheta L^\tau_n \vartheta^{-1} f$$

and L^τ_n denotes the usual Lagrange interpolation operator w.r.t. the nodes x^τ_{nk}, $k = 1, \ldots, n$. Moreover, $\operatorname{im} L_n$ is the image space of the orthoprojection L_n : $\mathbf{L}^2_\mu \longrightarrow \mathbf{L}^2_\mu$, $u \mapsto \sum_{k=0}^{n-1} \langle u, \widetilde{u}_k\rangle_\mu \widetilde{u}_k$ with the complete orthonormal system $\{\widetilde{u}_n\}_{n=0}^{\infty}$ in $\mathbf{L}^2_\mu$, where $\widetilde{u}_n(x) = \nu(x)p^\nu_n(x)$ in case $\vartheta = \nu$ and $\widetilde{u}_n(x) = v^{-\frac{1}{2},0}(x)p^\sigma_n(x)$ in case $\vartheta = v^{-\frac{1}{2},0}$.

Remark 3.1. *In* [5, 6, 7, 8] *the stability of collocation methods analogous to* (3.2) *is studied in spaces* $\mathbf{L}^2_{\alpha,\beta}$, *where weighted interpolation operators* $\vartheta L^\tau_n \vartheta^{-1} I$ *with more general* $\vartheta = v^{\alpha_0,\beta_0}$ *are used. The parameters* α_0 *and* β_0 *have to satisfy the conditions*

$$\alpha_0 = \frac{1}{4} - \frac{\alpha}{2} \neq 0\,, \quad \beta_0 = \frac{1}{4} - \frac{\beta}{2} \neq 0\,, \quad \alpha, \beta \in (-1, 1)\,, \tag{3.3}$$

which imply $-\frac{1}{4} < \alpha_0, \beta_0 < \frac{3}{4}$. *Since in the present situation* $\alpha_0 = -\frac{1}{2}$, *the results in* [5, 6, 7, 8] *do not cover the collocation methods* (3.2) *of interest in the present situation. Note also, that in case* $\vartheta(x) = \dfrac{1}{\sqrt{1-x}}$ *one has* $\beta_0 = 0$.

In the present paper we only pay attention to the computational aspects of the proposed collocation methods and the presentation of numerical results. The theoretical investigation of stability and convergence, which are rather technical (cf., for example, [5, 6, 7, 8]), keeps reserved to a forthcoming paper.

4. Computational aspects

Let us discuss computational aspects of the collocation methods (3.2) applied to (1.8) and (2.4). We are especially interested in an efficient assembling of the matrix entries of the collocation equations or, more precisely, in a fast matrix vector multiplication to be able to realize iterative methods for the solution of these equations effectively. For this end, we are looking for structural properties of the system matrices.

4.1. Collocation for equation (1.8)

Write the operator B of (1.8) in the form $B = S - N_0 + 6\,N_1 - 4\,N_2$, where, for $x \in (-1,1)$,

$$(Su)(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{u(y)}{y-x}\,dy \quad \text{and} \quad (N_\kappa u)(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{(1+x)^\kappa u(y)}{(2+y+x)^{\kappa+1}}\,dy\,.$$

Moreover, let us compute

$$H(x) = \int_{-1}^{1} \frac{1}{y-x}\,\frac{dy}{\sqrt{1-y}} \quad \text{for} \quad x \in (-\infty,1)\setminus\{-1\}\,.$$

Using the substitution $z^2 = 1-y$ we get

$$H(x) = 2\int_0^{\sqrt{2}} \frac{dz}{1-x-z^2} = \frac{1}{\sqrt{1-x}}\,\ln\frac{\sqrt{2}+\sqrt{1-x}}{\left|\sqrt{2}-\sqrt{1-x}\right|}\,. \tag{4.1}$$

Now, our aim is to compute the entries $\alpha_{jk}^\tau = \left(A\widetilde{\ell}_{nk}^\tau\right)(x_{nj}^\tau)$ of the matrices $\mathbb{A}_n$ associated to the collocation methods (3.2). First, let us look for representations of the matrices

$$\mathbb{S}_n^\tau = \left[\ \left(S\widetilde{\ell}_{nk}^\tau\right)(x_{nj}^\tau)\ \right]_{j,k=1}^n \quad \text{and} \quad \mathbb{N}_{\kappa,n}^\tau = \left[\ \left(N_\kappa\widetilde{\ell}_{nk}^\tau\right)(x_{nj}^\tau)\ \right]_{j,k=1}^n\,,$$

which admit to solve the linear systems

$$\mathbb{A}_n\xi_n = \eta_n\,, \quad \xi_n = \left[\ \xi_{nk}\ \right]_{k=1}^n\,,\ \eta_n = \left[\ f(x_{nk}^\tau)\ \right]_{k=1}^n\,, \tag{4.2}$$

corresponding to the collocation methods (3.2), in a fast manner.

Remark 4.1. *In the situations, under consideration here, the basis* $\left\{\widetilde{\ell}_{nk}^\tau\right\}_{k=1}^n$ *of the space* $\operatorname{im} L_n$ *is not orthonormal. Thus, stability of the collocation methods would not imply the uniform boundedness of the condition numbers of the sequence of matrices* $\{\mathbb{A}_n\}_{n=1}^\infty$ *. On the other hand, in case* $\tau = \sigma$ *one can show with the help of the Gaussian rule (w.r.t. the Chebyshev weight of first kind) that the system* $\left\{\sqrt{\frac{n}{\pi}}(1-x_{nk}^\sigma)^{-\frac{1}{2}}\widetilde{\ell}_{nk}^\tau\right\}_{k=1}^n$ *forms an orthonormal basis of* $\operatorname{im} L_n$ *. In case* $\tau = \varphi$ *this basis is "almost orthonormal". Indeed, for example, in case* $\vartheta = \nu$ *using the abbreviation*

$$\mu_{jk}^{(n)} = \sqrt{\frac{1-x_{nk}^\varphi}{1+x_{nk}^\varphi}}\sqrt{\frac{1-x_{nj}^\varphi}{1+x_{nj}^\varphi}} = \mu(x_{nk}^\varphi)\mu(x_{nj}^\varphi)\,,$$

we have

$$
\begin{aligned}
&\left\langle \widetilde{\ell}^{\varphi}_{nk}, \widetilde{\ell}^{\varphi}_{nj} \right\rangle_{\mu} \\
= \;& \mu^{(n)}_{jk} \int_{-1}^{1} \ell^{\varphi}_{nk}(x)\ell^{\varphi}_{nj}(x)\frac{\varphi(x)}{1-x}\,dx \\
= \;& \mu^{(n)}_{jk} \left[\int_{-1}^{1} \frac{\ell^{\varphi}_{nk}(x)\ell^{\varphi}_{nj}(x) - \ell^{\varphi}_{nk}(1)\ell^{\varphi}_{nj}(1)}{1-x}\varphi(x)dx + \pi \ell^{\varphi}_{nk}(1)\ell^{\varphi}_{nj}(1) \right] \\
= \;& \mu^{(n)}_{jk} \left[\sum_{m=1}^{n} \frac{\ell^{\varphi}_{nk}(x^{\varphi}_{nm})\ell^{\varphi}_{nj}(x^{\varphi}_{nm}) - \ell^{\varphi}_{nk}(1)\ell^{\varphi}_{nj}(1)}{1-x^{\varphi}_{nm}} \, \frac{\pi[\varphi(x^{\varphi}_{nm})]^2}{n+1} + \pi \ell^{\varphi}_{nk}(1)\ell^{\varphi}_{nj}(1) \right] \\
= \;& \frac{\pi\delta_{jk}(1-x^{\varphi}_{nk})}{n+1} + \pi\mu^{(n)}_{jk}\ell^{\varphi}_{nk}(1)\ell^{\varphi}_{nj}(1) \left[1 - \frac{1}{n+1}\sum_{m=1}^{n}(1+x^{\varphi}_{nm}) \right] \\
= \;& \frac{\pi}{n+1}\left[\delta_{jk}(1-x^{\varphi}_{nk}) + (-1)^{j+k}\varphi(x^{\varphi}_{nk})\varphi(x^{\varphi}_{nj}) \right] .
\end{aligned}
$$

Consequently, system (4.2) *can be preconditioned by diagonal matrices and we will solve the system*

$$\widetilde{\mathbb{A}}_n \widetilde{\xi}_n = \widetilde{\eta}_n \, , \tag{4.3}$$

where $\widetilde{\mathbb{A}}_n = \mathbb{D}_n \mathbb{A}_n \mathbb{D}_n^{-1}$, $\mathbb{D}_n = \operatorname{diag}\left[\sqrt{1-x^{\tau}_{nj}} \right]_{j=1}^{n}$, $\widetilde{\eta}_n = \mathbb{D}_n \eta_n$, *and* $\widetilde{\xi}_n = \mathbb{D}_n \xi_n$.

We will use the well-known relations

$$S\varphi U_k = -T_{k+1} \, , \quad S\sigma T_k = U_{k-1} \quad k = 0, 1, 2, \ldots \, , \tag{4.4}$$

and, for $0 \le k < n$,

$$\widehat{U}_n(x)U_k(x) = \frac{T_{n-k}(x) - T_{n+k+2}(x)}{2(1-x^2)} \, , \tag{4.5}$$

$$\widehat{U}_n(x)T_{k+1}(x) = \frac{1}{2}\left[U_{n-k-1}(x) + U_{n+k+1}(x) \right] , \tag{4.6}$$

$$\widehat{T}_n(x)T_k(x) = \frac{\sqrt{\delta^{\sigma}_k}}{2}\left[T_{n+k}(x) + T_{n-k}(x) \right] , \tag{4.7}$$

$$\widehat{T}_n(x)U_{k-1}(x) = \frac{1}{2}\left[U_{n+k-1}(x) - U_{n-k-1}(x) \right] . \tag{4.8}$$

where $\delta_0^\sigma = \frac{1}{2}$, $\delta_j^\sigma = 1$, $j = 1, 2, \ldots$, $\delta_j^\varphi = 1$, $j = 0, 1, 2, \ldots$ Define

$$h_n(x) = h_n^\tau(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{p_n^\tau(y)}{y - x} \vartheta(y)\, dy = \sqrt{\frac{2\delta_n^\tau}{\pi}}\, \widehat{h}_n^\tau(x) , \quad x \in \mathbb{R} \setminus \{\pm 1\} ,$$

and let us represent the weighted fundamental Lagrange interpolation polynomials in the form

$$\widetilde{\ell}_{nk}^\tau = \vartheta \sum_{m=1}^{n} \varepsilon_{mk}^\tau p_{m-1}^\tau \quad \text{with} \quad \varepsilon_{mk}^\tau = \left\langle \widetilde{\ell}_{nk}^\tau, \vartheta p_{m-1}^\tau \right\rangle_{\vartheta^{-2}\tau} . \tag{4.9}$$

Then,

$$\mathbb{S}_n^\tau = \mathbb{H}_n^\tau \mathbb{J}_n^\tau \quad \text{and} \quad \mathbb{N}_{\kappa,n}^\tau = (\mathbb{D}_n^\tau)^\kappa\, \mathbb{H}_{<,n}^{\tau,\kappa} \mathbb{J}_n^\tau , \tag{4.10}$$

where $\mathbb{J}_n^\tau = \left[\ \varepsilon_{mk}^\tau\ \right]_{m,k=1}^{n}$, $\mathbb{H}_n^\tau = \left[\ h_{m-1}(x_{nj}^\tau)\ \right]_{j,m=1}^{n}$, and

$$\mathbb{D}_n^\tau = \operatorname{diag}\left[\ (1 + x_{nk}^\tau)\ \right]_{k=1}^{n} , \quad \mathbb{H}_{<,n}^{\tau,\kappa} = \frac{1}{\kappa!} \left[\ h_{m-1}^{(\kappa)}(-2 - x_{nj}^\tau)\ \right]_{j,m=1}^{n} .$$

We get, using the Gaussian rules w.r.t. the weights $\sigma(x)$ and $\varphi(x)$,

$$\varepsilon_{mk}^\sigma = \frac{\sqrt{2\delta_{m-1}^\sigma \pi}}{n\, \vartheta(x_{nk}^\sigma)} \cos \frac{(m-1)(2k-1)\pi}{2n} \quad \text{and} \quad \varepsilon_{mk}^\varphi = \frac{\sqrt{2\pi}}{n+1} \frac{\varphi(x_{nk}^\varphi)}{\vartheta(x_{nk}^\varphi)} \sin \frac{mk\pi}{n+1} .$$

Consequently,

$$\mathbb{J}_n^\sigma = \frac{\sqrt{2\pi}}{n} \mathbb{E}_n \mathbb{C}_n^2 \mathbb{F}_{n,\sigma}^{\vartheta^{-1}} \quad \text{and} \quad \mathbb{J}_n^\varphi = \frac{\sqrt{2\pi}}{n+1} \mathbb{S}_n^1 \mathbb{F}_{n,\varphi}^{\vartheta^{-1}\varphi} , \tag{4.11}$$

where $\mathbb{E}_n = \operatorname{diag}\left[\ \sqrt{\delta_{k-1}^\sigma}\ \right]_{k=1}^{n}$, $\mathbb{F}_{n,\tau}^\chi = \operatorname{diag}\left[\ \chi(x_{nk}^\tau)\ \right]_{k=1}^{n}$, and $\mathbb{S}_n^1$ as well as $\mathbb{C}_n^2$ denote the first discrete sine and the second discrete cosine transforms,

$$\mathbb{S}_n^1 = \left[\ \sin \frac{jk\pi}{n+1}\ \right]_{j,k=1}^{n} \quad \text{and} \quad \mathbb{C}_n^2 = \left[\ \cos \frac{(j-1)(2k-1)\pi}{2n}\ \right]_{j,k=1}^{n} ,$$

respectively. This enables us to apply the matrices $\mathbb{J}_n^\tau$ to a vector of length n with $O(n \log n)$ complexity. If we set

$$\widehat{\mathbb{H}}_n^\tau = \left[\ \widehat{h}_{m-1}(x_{nj}^\tau)\ \right]_{j,m=1}^{n} \quad \text{and} \quad \widehat{\mathbb{H}}_{<,n}^{\tau,\kappa} = \frac{1}{\kappa!} \left[\ \widehat{h}_{m-1}^{(\kappa)}(-2 - x_{nj}^\tau)\ \right]_{j,m=1}^{n}$$

then

$$\mathbb{H}_n^\sigma = \sqrt{\frac{2}{\pi}}\, \widehat{\mathbb{H}}_n^\sigma \mathbb{E}_n , \quad \mathbb{H}_{<,n}^{\sigma,\kappa} = \sqrt{\frac{2}{\pi}}\, \widehat{\mathbb{H}}_{<,n}^{\sigma,\kappa} \mathbb{E}_n , \quad \mathbb{H}_n^\varphi = \sqrt{\frac{2}{\pi}}\, \widehat{\mathbb{H}}_n^\varphi , \quad \mathbb{H}_{<,n}^{\varphi,\kappa} = \sqrt{\frac{2}{\pi}}\, \widehat{\mathbb{H}}_{<,n}^{\varphi,\kappa} .$$

Let us explain the structure of the matrices $\mathbb{S}_n^\tau$ in the two different cases of the weight $\vartheta(x)$.

Case $\vartheta(x) = \sqrt{\frac{1+x}{1-x}}$: In view of (4.4) we get

$$\widehat{h}_{m-1}^\sigma(x) = (1 + x)\widehat{U}_{m-2}(x) + \delta_{m-1,0} ,$$

so that

$$\widehat{\mathbb{H}}_n^\sigma = \left[\ \nu(x_{nj}^\sigma) \sin \frac{(m-1)(2j-1)\pi}{2n} + \delta_{m-1,0}\ \right]_{j,m=1}^n = \mathbb{F}_{n,\sigma}^\nu \mathbb{S}_n^3 \mathbb{V}_n^T + \mathbb{P}_1 \,,$$

where by $\mathbb{S}_n^3 = (\mathbb{S}_n^2)^T$ we refer to the third discrete sine transform

$$\mathbb{S}_n^3 = \left[\ \sin \frac{k(2j-1)}{2n}\ \right]_{j,k=1}^n ,$$

by $\mathbb{V}_n$ to the matrix of the forward shift in $\mathbb{C}^n$, and by $\mathbb{P}_1$ to the matrix

$$\mathbb{P}_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} .$$

Together with the first relation in (4.11) we conclude

$$\mathbb{S}_n^\sigma = \mathbb{H}_n^\sigma \mathbb{J}_n^\sigma = \frac{2}{n} \left[\mathbb{F}_{n,\sigma}^\nu \mathbb{S}_n^3 \mathbb{V}_n^T + \mathbb{P}_1\right] \mathbb{E}_n^2 \mathbb{C}_n^2 \mathbb{F}_{n,\sigma}^\mu = \frac{2}{n} \left[\mathbb{F}_{n,\sigma}^\nu \mathbb{S}_n^3 \mathbb{V}_n^T + \frac{1}{2}\,\mathbb{P}_1\right] \mathbb{C}_n^2 \mathbb{F}_{n,\sigma}^\mu \,.$$

To get a representation for $\mathbb{H}_n^\varphi$ we make the following observation. For general $\vartheta(x)$, write

$$\begin{aligned} \widehat{h}_n^\varphi(x) \quad &= \quad \frac{1}{\pi} \int_{-1}^1 \frac{\widehat{U}_n(y) - \widehat{U}_n(x)}{y-x}\, \vartheta(y)\, dy + \frac{\widehat{U}_n(x)}{\pi} \int_{-1}^1 \frac{\vartheta(y)}{y-x}\, dy \\ &=: \quad g_n^\varphi(x) + \widehat{U}_n(x) \widehat{h}_0(x) \end{aligned}$$

and

$$g_n^\varphi(x) = \sum_{k=0}^{n-1} \gamma_{nk}^\varphi U_k(x) \quad \text{with} \quad \gamma_{nk}^\varphi = \int_{-1}^1 g_n^\varphi(x) U_k(x) \varphi(x)\, dx \,.$$

We have

$$\gamma_{nk}^\varphi = \int_{-1}^1 \frac{1}{\pi} \int_{-1}^1 \frac{\widehat{U}_n(y) - \widehat{U}_n(x)}{y-x}\, U_k(x) \varphi(x)\, dx\, \vartheta(y)\, dy =: \int_{-1}^1 Q_{nk}^\varphi(y) \vartheta(y)\, dy \,.$$

Using (4.4), (4.5), and (4.6) we get

$$Q_{nk}^\varphi(y) = \widehat{U}_n(y) T_{k+1}(y) - \frac{1}{2\pi} \int_{-1}^1 \frac{T_{n-k}(x) - T_{n+k+2}(x)}{y-x}\, \sigma(x)\, dx = U_{n-k-1}(y) \,.$$

It follows

$$\gamma_{nk}^\varphi = \int_{-1}^1 U_{n-k-1}(y) \vartheta(y)\, dy =: \gamma_{n-k-1} \,, \quad k = 0, 1, \ldots, n-1 \,. \tag{4.12}$$

Set $\widetilde{\gamma}_j = 0$ if $j < 0$, $\widetilde{\gamma}_j = \sqrt{\frac{2}{\pi}}\,\gamma_j$ if $j \geq 0$, and let

$$\mathbf{\Gamma}_n = \left[\ \widetilde{\gamma}_{m-k-1}\ \right]_{k,m=0}^{n-1} = \begin{bmatrix} 0 & \widetilde{\gamma}_0 & \widetilde{\gamma}_1 & \cdots & \widetilde{\gamma}_{n-2} \\ & 0 & \widetilde{\gamma}_0 & \cdots & \widetilde{\gamma}_{n-3} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \widetilde{\gamma}_0 \\ & & & & 0 \end{bmatrix}.$$

Then, in the present case ($\vartheta = \nu$, $\tau = \varphi$) we get

$$\begin{aligned} \widehat{\mathbb{H}}_n^{\varphi} &= \left[\ \sum_{k=0}^{m-1} \widetilde{\gamma}_{m-k-2}\widehat{U}_k(x_{nj}^{\varphi})\ \right]_{j,m=1}^{n} + \left[\ \widehat{U}_{m-1}(x_{nj}^{\varphi})\widehat{h}_0(x_{nj}^{\varphi})\ \right]_{j,m=1}^{n} \\ &= \mathbb{F}_{n,\varphi}^{\sigma}\left[\mathbb{S}_n^1\mathbf{\Gamma}_n + \mathbb{S}_n^1\right], \end{aligned}$$

where we took into account that $\widehat{h}_0(x) = 1$, $-1 < x < 1$. Consequently,

$$S_n^{\varphi} = \mathbb{H}_n^{\varphi}\mathbb{J}_n^{\varphi} = \frac{2}{n+1}\,\mathbb{F}_{n,\varphi}^{\sigma}\mathbb{S}_n^1\left[\mathbf{\Gamma}_n + \mathbb{I}_n\right]\mathbb{S}_n^1\mathbb{F}_{n,\varphi}^{p_1},$$

where $p_1(x) = 1 - x$. To compute the γ_k's in this case we define

$$\widehat{\gamma}_k = \int_{-1}^{1} \widehat{U}_k(x)\nu(x)\,dx$$

and use the relations

$$\widehat{\gamma}_k = \int_{-1}^{1}(1+x)\widehat{U}_k(x)\,\frac{dx}{\sqrt{1-x^2}} = \begin{cases} \displaystyle\int_{-1}^{1}\widehat{U}_k(x)\,\frac{dx}{\sqrt{1-x^2}} & : \quad k \ \text{ even}, \\ \displaystyle\int_{-1}^{1}x\widehat{U}_k(x)\,\frac{dx}{\sqrt{1-x^2}} & : \quad k \ \text{ odd}, \end{cases}$$

and

$$\widehat{U}_{2k}(x) = 2\sum_{m=0}^{k}\widehat{T}_{2m}(x) - 1, \quad \widehat{U}_{2k-1} = 2\sum_{m=0}^{k-1}\widehat{T}_{2m+1}(x) \tag{4.13}$$

to get

$$\widehat{\gamma}_{2k} = \int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = \pi \quad \text{and} \quad \widehat{\gamma}_{2k-1} = \int_{-1}^{1}\frac{2x^2\,dx}{\sqrt{1-x^2}} = \pi,$$

so that $\widetilde{\gamma}_k = 2$, $k = 0, 1, 2, \ldots$.

Case $\vartheta(x) = \frac{1}{\sqrt{1-x}}$: Analogously to (4.12) we get

$$\begin{aligned}\widehat{h}_n^\sigma(x) &= \frac{1}{\pi}\int_{-1}^1 \frac{\widehat{T}_n(y)-\widehat{T}_n(x)}{y-x}\,\vartheta(y)\,dy + \frac{\widehat{T}_n(x)}{\pi}\int_{-1}^1 \frac{\vartheta(y)}{y-x}\,dy \\ &=: g_n^\sigma(x) + \widehat{T}_n(x)\widehat{h}_0(x)\,,\end{aligned}$$

and

$$g_n^\sigma(x) = \sum_{k=0}^{n-1}\gamma_{nk}^\sigma T_n(x) \quad \text{with} \quad \gamma_{nk}^\sigma = \int_{-1}^1 g_n^\sigma(x)T_k(x)\sigma(x)\,dx\,,$$

$$\gamma_{nk}^\sigma = \int_{-1}^1 \frac{1}{\pi}\int_{-1}^1 \frac{\widehat{T}_n(y)-\widehat{T}_n(x)}{y-x}\,T_k(x)\sigma(x)\,dx\,\vartheta(y)\,dy =: \int_{-1}^1 Q_{nk}^\sigma(y)\vartheta(y)\,dy\,.$$

Using (4.4), (4.7), and (4.8) we conclude

$$\begin{aligned}Q_{nk}^\sigma(y) &= -\widehat{T}_n(y)U_{k-1}(y) - \frac{\sqrt{\delta_k^\sigma}}{2\pi}\int_{-1}^1 \frac{T_{n+k}(x)+T_{n-k}(x)}{y-x}\,\sigma(x)\,dx \\ &= \sqrt{\delta_k^\sigma}\,U_{n-k-1}(y)\,,\end{aligned}$$

so that

$$\gamma_{nk}^\sigma = \sqrt{\delta_k^\sigma}\int_{-1}^1 U_{n-k-1}(y)\vartheta(y)\,dy = \sqrt{\delta_k^\sigma}\,\gamma_{n-k-1}\,, \quad k = 0,1,\ldots,n-1\,. \tag{4.14}$$

With the help of (4.13) we find in this case ($\vartheta = v^{-\frac{1}{2},0}$)

$$\widetilde{\gamma}_{2k} = \frac{4\sqrt{2}}{\pi}\left[\sum_{m=1}^k\left(\frac{1}{4m+1}-\frac{1}{4m-1}\right)+1\right], \tag{4.15}$$

$$\widetilde{\gamma}_{2k+1} = \frac{4\sqrt{2}}{\pi}\sum_{m=0}^k\left(\frac{1}{4m+1}-\frac{1}{4m+3}\right). \tag{4.16}$$

Thus,

$$\widehat{\mathbb{H}}_n^\sigma = \left[\ \sum_{k=0}^{m-1}\widetilde{\gamma}_{m-k-2}\widehat{T}_k(x_{nj}^\sigma)\delta_k^\sigma\ \right]_{j,k=1}^n + \left[\ \widehat{T}_{m-1}(x_{nj}^\sigma)\widehat{h}_0(x_{nj}^\sigma)\ \right]_{j,m=1}^n,$$

and, taking into account the first relation in (4.11), $\mathbf{\Gamma}_n\mathbb{E}_n^2 = \mathbf{\Gamma}_n$, and $\mathbb{C}_n^3\mathbb{E}_n^2\mathbb{C}_n^2 = \frac{n}{2}\,\mathbb{I}_n$, where $\mathbb{C}_n^3 = \left(\mathbb{C}_n^2\right)^T$ is the third discrete cosine transform, we arrive at

$$\mathbb{S}_n^\sigma = \frac{2}{n}\left[\mathbb{C}_n^3\mathbb{E}_n^2\mathbf{\Gamma}_n + \mathbb{D}_n^\sigma\mathbb{C}_n^3\right]\mathbb{E}_n^2\mathbb{C}_n^2\mathbb{F}_{n,\sigma}^{\vartheta^{-1}\varphi} = \left[\frac{2}{n}\,\mathbb{C}_n^3\mathbb{E}_n^2\mathbf{\Gamma}_n\mathbb{C}_n^2 + \mathbb{D}_n^\sigma\right]\mathbb{F}_{n,\sigma}^{\vartheta^{-1}\varphi}\,,$$

where $\mathbb{D}_n^\sigma = \operatorname{diag}\left[\ \widehat{h}_0(x_{nj}^\sigma)\ \right]_{j=1}^n$ and (cf. (4.1))

$$\widehat{h}_0(x) = \frac{2}{\pi\sqrt{1-x}}\,\ln\frac{\sqrt{2}+\sqrt{1-x}}{\sqrt{1+x}}\,.$$

For the Chebyshev nodes of second kind, we have

$$\begin{aligned}\widehat{\mathbb{H}}_n^\sigma &= \left[\ \sum_{k=0}^{m-1} \widetilde{\gamma}_{m-k-2}\widehat{U}_k(x_{nj}^\varphi)\ \right]_{j,m=1}^n + \left[\ \widehat{U}_{m-1}(x_{nj}^\varphi)\widehat{h}_0(x_{nj}^\varphi)\ \right]_{j,m=1}^n \\ &= \mathbb{F}_{n,\varphi}^\sigma \left[\mathbb{S}_n^1 \mathbf{\Gamma}_n + \mathbb{D}_n^\varphi \mathbb{S}_n^1\right] ,\end{aligned}$$

so that, taking into account the second relation in (4.11) and $\mathbb{S}_n^1\mathbb{S}_n^1 = \frac{n+1}{2}\,\mathbb{I}_n$,

$$\mathbb{S}_n^\varphi = \mathbb{F}_{n,\varphi}^\sigma \left[\frac{2}{n+1}\,\mathbb{S}_n^1\mathbf{\Gamma}_n\mathbb{S}_n^1 + \mathbb{D}_n^\varphi\right]\mathbb{F}_{n,\varphi}^{\vartheta^{-1}\varphi} = \frac{2}{n+1}\,\mathbb{F}_{n,\varphi}^\sigma\mathbb{S}_n^1\mathbf{\Gamma}_n\mathbb{S}_n^1 + \mathbb{F}_{n,\varphi}^{\vartheta^{-1}}\mathbb{D}_n^\varphi\,.$$

In order to realize the matrix vector multiplication with the matrices $\mathbb{H}_{<,n}^{\tau,\kappa}$ without saving their entries by at most $O(n^2)$ computational complexity we follow an idea given in [13]. Due to the recurrence relations

$$\widehat{T}_{n+1}(x) = 2x\,\widehat{T}_n(x) - \widehat{T}_{n-1}(x)\,, \quad n = 1,2,\dots\,, \qquad \widehat{T}_1(x) = x\,\widehat{T}_0(x)\,, \tag{4.17}$$

and

$$\widehat{U}_{n+1}(x) = 2x\,\widehat{U}_n(x) - \widehat{U}_{n-1}(x)\,, \quad n = 0,1,2,\dots\,,\ \widehat{U}_{-1} \equiv 0\,, \tag{4.18}$$

we have

$$\widehat{h}_{n-1}^\sigma(x) - 2x\,\widehat{h}_n^\sigma(x) + \widehat{h}_{n+1}^\sigma(x) = \frac{2}{\pi}\int_{-1}^1 \widehat{T}_n(y)\vartheta(y)\,dy =: \gamma_n^\sigma\,, \quad n = 1,2,\dots\,,$$

$$-x\,\widehat{h}_0^\sigma(x) + \widehat{h}_1^\sigma(x) = \frac{1}{\pi}\int_{-1}^1 \widehat{T}_0(y)\vartheta(y)\,dy =: \gamma_0^\sigma\,,$$

$$\widehat{h}_{n-1}^\varphi(x) - 2x\,\widehat{h}_n^\varphi(x) + \widehat{h}_{n+1}^\varphi(x) = \frac{2}{\pi}\int_{-1}^1 \widehat{U}_n(y)\vartheta(y)\,dy =: \gamma_n^\varphi\,, \quad n = 0,1,2,\dots\,,$$

and

$$\widehat{h}_{n-1}^{(\kappa)}(x) - 2x\,\widehat{h}_n^{(\kappa)}(x) + \widehat{h}_{n+1}^{(\kappa)}(x) = 2\delta_n^\tau\kappa\,\widehat{h}_n^{(\kappa-1)}(x)\,, \quad \kappa = 1,2,\dots\,.$$

Thus, to compute the entries of $\mathbb{H}_{<,n}^{\tau,\kappa}$ we have to solve the linear systems

$$\begin{bmatrix} -2\delta_0^\tau z & 1 & & & \\ 1 & -2z & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2z & 1 \\ & & & 1 & -2z \end{bmatrix} \begin{bmatrix} \widehat{h}_0(z) \\ \widehat{h}_1(z) \\ \vdots \\ \widehat{h}_{n-3}(z) \\ \widehat{h}_{n-2}(z) \end{bmatrix} = \begin{bmatrix} \gamma_0^\tau \\ \gamma_1^\tau \\ \vdots \\ \gamma_{n-3}^\tau \\ \gamma_{n-2}^\tau - \widehat{h}_{n-1}(z) \end{bmatrix}$$

and, for $\kappa = 1, 2, \dots$,

$$\begin{bmatrix} -2\delta_0^\tau z & 1 & & & \\ 1 & -2z & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2z & 1 \\ & & & 1 & -2z \end{bmatrix} \begin{bmatrix} \widehat{h}_0^{(\kappa)}(z) \\ \widehat{h}_1^{(k)}(z) \\ \vdots \\ \widehat{h}_{n-3}^{(\kappa)}(z) \\ \widehat{h}_{n-2}^{(\kappa)}(z) \end{bmatrix} = \begin{bmatrix} 2\kappa\delta_0^\tau \widehat{h}_0^{(\kappa-1)}(z) \\ 2\kappa \widehat{h}_1^{(\kappa-1)}(z) \\ \vdots \\ 2\kappa \widehat{h}_{n-3}^{(\kappa-1)}(z) \\ 2\kappa \widehat{h}_{n-2}^{(\kappa-1)}(z) - \widehat{h}_{n-1}^{(\kappa)}(z) \end{bmatrix}$$

for $z = -2 - x_{nj}^\tau$, $j = 1, \dots, n$. For doing this in a stable manner the following lemma is important, a proof of which one can found in [13, Sect.s 6.3.2 and 8].

Lemma 4.2. *For $z = -2 - x_{nj}^\tau$, $j = 1, \dots, n$, the condition numbers of the matrices*

$$\operatorname{tridiag} \left[\begin{array}{ccc} 1 & -2\delta_n^\tau z & 1 \end{array} \right]$$

of order n are uniformly bounded w.r.t. n .

Of course, the values $h_{n-1}^{(\kappa)}(-2 - x_{nj}^\tau)$ have to be precomputed, for example, with the help of appropriate quadrature rules. For instance, the values $h_{n-1}(x) = h_{n-1}^\sigma(x)$, $x < -1$, we can approximate by

$$\frac{1}{N} \sum_{k=1}^{N} \frac{\left(1 + \cos\frac{(2k-1)\pi}{2N}\right) \cos\frac{(n-1)(2k-1)\pi}{2N}}{\cos\frac{(2k-1)\pi}{2N} - x}$$

with N sufficiently large. In [4] there will be discussed the application of fast summation methods to realize the matrix vector multiplication with the matrices $\mathbb{H}_{<,n}^{\tau,\kappa}$.

In case $\vartheta = \nu$ we have

$$\gamma_0^\sigma = \gamma_1^\sigma = 1 \quad \text{and} \quad \gamma_n^\sigma = 0\,, \; n = 2, 3, \dots$$

and (cf. (4.12))

$$\gamma_n^\sigma = \widetilde{\gamma}_n = 2\,, \quad n = 0, 1, 2, \dots .$$

In case $\vartheta = v^{-\frac{1}{2},0}$ we get

$$\gamma_n^\sigma = \frac{\sqrt{2}(-1)^{n+1}\delta_n^\sigma}{\pi\left(n^2 - \frac{1}{4}\right)} \quad \text{and} \quad \gamma_n^\varphi = \widetilde{\gamma}_n$$

with $\widetilde{\gamma}_n$ from (4.15) and (4.16).

4.2. Collocation for equation (2.4)

For this equation we fix the weight $\vartheta(x) = \nu(x) = \sqrt{\frac{1+x}{1-x}}$. For the matrix

$$\mathbb{A}_n = \left[\begin{array}{c} \left(A\widetilde{\ell}_{nk}^\tau\right)(x_{nj}^\tau) \end{array} \right]_{j,k=1}^n$$

we can use the relation $A\widetilde{\ell}_{nk}^\tau = v^{0,\frac{1}{2}} B v^{0,-\frac{1}{2}} \widetilde{\ell}_{nk}^\tau$, where B is the operator of equation (1.8) (cf. also (2.3)). Hence, $\mathbb{A}_n = \mathbb{D}_n \mathbb{B}_n \mathbb{D}_n^{-1}$, where

$$\mathbb{B}_n = \left[\begin{array}{c} \left(B\widetilde{\ell}_{nk}^\tau\right)(x_{nj}^\tau) \end{array} \right]_{j,k=1}^n$$

is the matrix representation of $M_n^\tau B L_n : \operatorname{im} L_n \longrightarrow \operatorname{im} L_n$ in case $\vartheta = v^{-\frac{1}{2},0}$. Moreover, $\mathbb{D}_n = \operatorname{diag}\left[\ v^{0,\frac{1}{2}}(x_{nj}^\tau)\ \right]_{j=1}^n$. Finally, w.r.t. the basis

$$\left\{ \sqrt{\frac{n}{\pi(1-x_{nk}^\tau)}}\, \widetilde{\ell}_{nk}^\tau : k = 1, \ldots, n \right\}$$

we get the representation

$$\widetilde{\mathbb{A}}_n = \widetilde{\mathbb{D}}_n \mathbb{B}_n \widetilde{\mathbb{D}}_n^{-1}$$

with $\widetilde{\mathbb{D}}_n = \operatorname{diag}\left[\ \varphi(x_{nj}^\tau)\ \right]_{j=1}^n$.

5. Numerical results

In this section we present the numerical results obtained by the application of the collocation methods (3.2) to equation (1.8) (in case $f(x) = -\frac{1+\kappa}{2\mu}\, p_0 \equiv 1$) with $\vartheta(x) = \nu(x)$ and $\vartheta(x) = v^{-\frac{1}{2},0}(x)$ and to the transformed equation (2.4) with $\vartheta(x) = \nu(x)$ (cf. Sections 4.1 and 4.2). The linear system (4.3) is solved iteratively by the Krylov subspace method CGNR (cf. [17] or [9]) using the function $v^{-\frac{1}{2},0}(x)$ in case of equation (1.8) and the function $\nu(x)$ in case of equation (2.4) as initial guess. The iteration is stopped when the (accumulated) residual norm is smaller than 10^{-12} times the initial residual norm. The following tables show the number M of iterations necessary to obtain this prescribed accuracy, the condition number $\operatorname{cond}(\widetilde{\mathbb{A}}_n)$ (in spectral norm) of the matrices of the systems (4.3), and the computed normalized stress intensity factor $\widetilde{k}_1(1) = \frac{k_1(1)}{p_0\sqrt{2}}$ (cf. (1.9)). This number is calculated by interpolating the approximate values of $u'(x)\sqrt{1-x}$ (obtained from (4.3)) at x_{n1}^τ, x_{n2}^τ, and x_{n3}^τ.

n	M	$\operatorname{cond}(\widetilde{\mathbb{A}}_n)$	$\operatorname{cond}(\mathbb{A}_n)$	$\widetilde{k}_1(1)$	$d(-1)$
8	8	1.32	13.5	1.1236085055	1.4526062574
16	9	1.35	27.1	1.1218585764	1.4542043666
32	9	1.37	54.1	1.1215858066	1.4543532905
64	10	1.38	108.1	1.1215356262	1.4543312321
128	10	1.40	216.1	1.1215252855	1.4543110592
256	10	1.41	432.2	1.1215229738	1.4543023509
512	11	1.41	864.3	1.1215224300	1.4542992576
1024	11	1.42	1728.5	1.1215222983	1.4542982552
2048	12	1.42	3456.9	1.1215222660	1.4542979473

Collocation for (1.8) with $\tau = \sigma$ and $\vartheta(x) = \sqrt{\frac{1+x}{1-x}}$

n	M	$\mathrm{cond}(\widetilde{\mathbb{A}}_n)$	$\mathrm{cond}(\mathbb{A}_n)$	$\widetilde{k}_1(1)$
7	9	8.6	15.86	1.1503638640
15	13	16.5	59.54	1.1275040512
31	17	32.4	233.97	1.1228513772
63	18	64.4	932.28	1.1218278789
127	21	128.4	$3.73 \cdot 10^3$	1.1215945578
255	23	256.4	$1.49 \cdot 10^4$	1.1215397220
511	25	512.4	$5.96 \cdot 10^4$	1.1215265338
1023	27	1024.4	$2.39 \cdot 10^5$	1.1215233124
2047	29	2048.4	$9.54 \cdot 10^5$	1.1215225178

Collocation for (1.8) with $\tau = \varphi$ and $\vartheta(x) = \sqrt{\frac{1+x}{1-x}}$

We remark that Koiter [12] computed $\widetilde{k}_1(1) = 1.12152226$ with the help of analytical methods (cf. also [11, Table 2 and Appendix D]). Moreover, we compare $\mathrm{cond}(\widetilde{\mathbb{A}}_n)$ with $\mathrm{cond}(\mathbb{A}_n)$ (see Remark 4.1), and in case $\tau = \sigma$, $\vartheta = \nu$ the normalized crack opening displacement $d(-1)$ at point $x = -1$ is computed,

$$d(-1) = \left(\frac{2\mu}{1+\kappa}\right)\frac{u(-1)}{2p_0}\,.$$

For this we use the relation

$$u(-1) \approx u_n(-1) = -\int_{-1}^{1} v_n(x)\,dx = -\frac{\pi}{n}\sum_{k=1}^{n} \varphi(x_{nk}^{\sigma})\xi_{nk}$$

if $v_n(x)$ is the solution of the respective collocation method for (1.8).

n	M	$\mathrm{cond}(\widetilde{\mathbb{A}}_n)$	$\mathrm{cond}(\mathbb{A}_n)$	$\widetilde{k}_1(1)$
8	8	3.73	9.5	1.1215270375
16	10	5.31	19.1	1.1215228492
32	10	7.53	38.2	1.1215222715
64	11	10.67	76.4	1.1215222553
128	12	15.11	152.8	1.1215222553
256	12	21.38	305.6	1.1215222552
512	13	30.24	611.1	1.1215222552
1024	13	42.78	1222.2	1.1215222552
2048	14	60.50	2444.4	1.1215222552

Collocation for (1.8) with $\tau = \sigma$ and $\vartheta(x) = \frac{1}{\sqrt{1-x}}$

n	M	cond($\widetilde{\mathbb{A}}_n$)	cond($\mathbb{A}_n$)	$\widetilde{k}_1(1)$
7	9	19.93	31.15	1.1221246506
15	13	76.15	209.65	1.1215271925
31	14	300.30	$1.59 \cdot 10^3$	1.1215222672
63	16	$1.20 \cdot 10^3$	$1.25 \cdot 10^4$	1.1215222567
127	18	$4.78 \cdot 10^3$	$9.99 \cdot 10^4$	1.1215222553
255	20	$1.91 \cdot 10^4$	$7.98 \cdot 10^5$	1.1215222552
511	22	$7.63 \cdot 10^4$	$6.39 \cdot 10^6$	1.1215222552
1023	25	$3.06 \cdot 10^5$	$5.11 \cdot 10^7$	1.1215222552
2047	29	$1.22 \cdot 10^6$	$4.09 \cdot 10^8$	1.1215222552

Collocation for (1.8) with $\tau = \varphi$ and $\vartheta(x) = \frac{1}{\sqrt{1-x}}$

n	M	cond($\widetilde{\mathbb{A}}_n$)	cond($\mathbb{A}_n$)	$\widetilde{k}_1(1)$	$d(-1)$
8	8	2.37	23.2	1.1215079459	1.4567943019
16	10	2.65	46.1	1.1215226201	1.4549287995
32	11	2.88	91.7	1.1215222682	1.4544556171
64	12	3.06	182.7	1.1215222553	1.4543372487
128	13	3.22	364.4	1.1215222552	1.4543076760
256	14	3.34	728.0	1.1215222552	1.4543002834
512	15	3.45	1455.1	1.1215222552	1.4542984353
1024	16	3.53	2909.3	1.1215222552	1.4542979733
2048	17	3.61	5817.7	1.1215222552	1.4542978577

Collocation for (2.4) with $\tau = \sigma$ and $\vartheta(x) = \sqrt{\frac{1+x}{1-x}}$

n	M	cond($\widetilde{\mathbb{A}}_n$)	cond($\mathbb{A}_n$)	$\widetilde{k}_1(1)$
7	9	9.6	29.50	1.1218787941
15	12	19.1	106.23	1.1215247000
31	13	37.6	401.97	1.1215222327
63	15	74.6	$1.56 \cdot 10^3$	1.1215222562
127	17	148.5	$6.14 \cdot 10^3$	1.1215222553
255	19	296.2	$2.43 \cdot 10^4$	1.1215222552
511	21	591.5	$6.69 \cdot 10^4$	1.1215222552
1023	22	1182.1	$3.86 \cdot 10^5$	1.1215222552
2047	24	2363.4	$1.54 \cdot 10^6$	1.1215222552

Collocation for (2.4) with $\tau = \varphi$ and $\vartheta(x) = \sqrt{\frac{1+x}{1-x}}$

We observe that the best results are obtained when the exact asymptotic of the solution is used, i.e., $\vartheta(x) = v^{-\frac{1}{2},0}(x)$ in case of equation (1.8) and $\vartheta(x) = \nu(x)$ in case of equation (2.4). Moreover, the condition numbers of $\widetilde{\mathbb{A}}_n$ are in any case much smaller than the condition numbers of $\mathbb{A}_n$, where the condition numbers for $\tau = \sigma$ are smaller than the condition numbers for $\tau = \varphi$. The condition numbers seem to be uniformly bounded (w.r.t. n) only in two cases, namely for $\tau = \sigma$, $\vartheta(x) = \nu(x)$, equation (1.8) and for $\tau = \sigma$, $\vartheta(x) = \nu(x)$, equation (2.4).

For comparison we present also the results of [11] concerning the same equation, which are obtained by solving the original hypersingular integral equation (1.1). The method of [11] consists in approximating the unknown function by a truncated weighted power series

$$\sqrt{1-x}\sum_{k=0}^{N} a_{nk}(x+1)^k$$

and determining the unknown coefficient a_{nk} by collocating w.r.t. the Chebyshev nodes of first kind of order $N+1$.

N	$\widetilde{k}_1(1)$	$d(-1)$
1	1.062652	1.502816
2	1.126950	1.423476
3	1.124283	1.457747
4	1.121818	1.457747
5	1.121442	1.454520
6	1.121251	1.454224
7	1.121483	1.454211
8	1.121504	1.454241
9	1.121514	1.454264
10	1.121518	1.454278
15	1.121522	1.454298
20	1.121522	1.454298

The results of [11, p. 116]

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Spectrum and Principal Function of Operators

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Abstract. Let $T = U|T|$ be an invertible operator with $|T| - |T^*| = \xi \otimes \xi$. Under the assumption that T is pure, we characterize the residual spectrum and the essential spectrum of T by the principal function.

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Keywords. Hilbert space, spectrum, principal function.

1. Introduction

K.F. Clancey studied operators T on a separable Hilbert space $\mathcal{H}$ whose self-commutators have the form

$$[T^*, T] = T^*T - TT^* = \xi \otimes \xi \tag{1}$$

for some ξ in $\mathcal{H}$. Since $\xi \otimes \xi \geq 0$, by (1) the operator T is hyponormal. In [9], spectral properties of a pure hyponormal operator T with one-dimensional self-commutator have been shown (see Chapter XI). Especially, the following theorems hold:

Theorem A ([9, Chapter XI, Theorem 4.1]). *Let $T = U|T|$ be a pure hyponormal operator with one-dimensional self-commutator. A point $z \in \mathbf{C}$ belongs to the residual spectrum of T if and only if, for a compact ball* B *centered at z,*

$$\iint_{\mathrm{B}} \frac{1 - g_T(x, y)}{|x + iy - z|^2}\, d\mu(x, y) \ < \ \infty,$$

where g_T is the principal function of T and μ is the Lebesgue measure on $\mathbf{C} = \mathbf{R}^2$.

Theorem B ([9, Chapter XI, Theorem 4.2]). *Let $T = U|T|$ be a pure hyponormal operator with one-dimensional self-commutator. A point $z \in \mathbf{C}$ belongs to the essential spectrum of T if and only if the principal function is not almost everywhere equal to* 0 *or* 1 *in a neighbourhood of z.*

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In this paper, we study theorems above in the case of an invertible operator T with

$$|T| - |T^*| = \xi \otimes \xi. \tag{2}$$

An operator below means a bounded linear operator on a separable infinite-dimensional Hilbert space $\mathcal{H}$. Let $\mathcal{C}_1$ be the set of all trace class operators. An operator T is said to be semi-hyponormal if $|T| \geq |T^*|$. Hence, if an operator T satisfies (2), then T is semi-hyponormal and $|T| - |T^*| \in \mathcal{C}_1$. Since $T^*T - TT^* = |T|(|T| - |T^*|) + (|T| - |T^*|)|T^*|$, in this case we have $[T^*, T] \in \mathcal{C}_1$. By Löwner's inequality, it holds that if T is hyponormal, then T is semi-hyponormal. There exists a semi-hyponormal operator which is not hyponormal. It is clear that if $T = U|T|$ is semi-hyponormal, then $S = U|T|^{1/2}$ is hyponormal.

Theorem C ([5, Lemma 1]). *Let $T = U|T|$ be the polar decomposition of an invertible operator T. Then $[T^*, T] \in \mathcal{C}_1$ if and only if $[|T|, U] \in \mathcal{C}_1$.*

For differentiable functions ϕ, ψ of two variables (x, y), we denote Jacobian of ϕ, ψ by $J(\phi, \psi)(x, y)$, i.e., $J(\phi, \psi)(x, y) = \dfrac{\partial \phi}{\partial x}\dfrac{\partial \psi}{\partial y} - \dfrac{\partial \phi}{\partial y}\dfrac{\partial \psi}{\partial x}$. $\mathrm{Tr}(\cdot)$ denotes the trace. For an operator $T = X + iY = U|T|$, we consider the following trace formulae:

$$\mathrm{Tr}([P(X,Y), Q(X,Y)]) = \frac{1}{2\pi i} \iint J(P,Q)(x,y) g_T(x,y)\, dxdy, \tag{3}$$

for polynomials P and Q.

$$\mathrm{Tr}([\phi(|T|,U), \psi(|T|,U)]) = \frac{1}{2\pi} \iint J(\phi,\psi)(r, e^{i\theta}) e^{i\theta} g_T^P(e^{i\theta}, r)\, drd\theta \tag{4}$$

for Laurent polynomials ϕ and ψ.

If formula (3) holds, the function g_T is called *the principal function related to the Cartesian decomposition* $T = X + iY$. If formula (4) holds, the function g_T^P is called *the principal function related to the polar decomposition* $T = U|T|$. For an invertible operator T such that $[T^*, T] \in \mathcal{C}_1$, there exist both g_T and g_T^P ([1], [3]).

Definition. Let $T = U|T|$ be semi-hyponormal and $S = U|T|^{1/2}$. Then we define the principal function $g_T^P(e^{i\theta}, r)$ of T by

$$g_T^P(e^{i\theta}, r) = g_S^P(e^{i\theta}, \sqrt{r}),$$

where is $g_S^P(e^{i\theta}, r)$ is the principal function of the hyponormal operator S.

The following result holds.

Theorem D ([4, Corollary 2]). *If an invertible operator $T = X + iY = U|T|$ satisfies $[|T|, U] \in \mathcal{C}_1$, then $g_T(x, y) = g_T^P(e^{i\theta}, r)$, where $x + iy = re^{i\theta}$.*

Theorem E ([3, Theorem 8]). *Let $T = X + iY = U|T|$ be a semi-hyponormal operator with $[|T|, U] \in \mathcal{C}_1$. Then there exist summable principal functions $g_T(x, y)$ and*

$g_T^P(e^{i\theta}, r)$ associated with the Cartesian decomposition and the polar decomposition of T,respectively, such that

$$g_T(x, y) = g_T^P(e^{i\theta}, r)$$

almost everywhere $x + iy = re^{i\theta}$ on $\mathbf{C}$.

In this paper, first we show the following

Theorem 1. *Let $T = U|T|$ be an invertible pure semi-hyponormal operator with $|T| - |T^*| = \xi \otimes \xi$. A point $z = ae^{i\eta} \in \mathbf{C}$ belongs to the residual spectrum of T if and only if, for a compact ball* B *centered at z,*

$$\iint_{\mathrm{B}} \frac{1 - g_T(a^{-1}\sqrt{x^2+y^2} \cdot x, a^{-1}\sqrt{x^2+y^2} \cdot y)}{|x+iy-z|^2}\, d\mu(x, y) \ < \ \infty,$$

where g_T is the principal function of T and μ is the Lebesgue measure on $\mathbf{C} = \mathbf{R}^2$.

2. Proof

First, we prepare some lemmas. We denote the spectrum, the residual spectrum and the essential spectrum of T by $\sigma(T)$, and $\sigma_r(T)$ and $\sigma_{ess}(T)$, respectively. Following results have been shown by [10, Chapter 6, Lemma 3.6] if U is unitary. Lemma 1 is easy to check.

Lemma 1. *Let $T = U|T|$ be an operator and put $S = U|T|^{1/2}$. Then the following statements hold.*

(1) *S is pure if and only T is pure.*
(2) $\ker S = \{0\}$ *if and only if* $\ker T = \{0\}$.
(3) $0 \in \sigma(S)$ *if and only* $0 \in \sigma(T)$.

For an operator A and a complex number λ with $\ker(A - \lambda I) = \{0\}$, there exists a positive number c such that $||(A - \lambda I)x|| \geq c||x||$ if and only if $(A - \lambda I)\mathcal{H}$ is closed.

Lemma 2. *Let $T = U|T|$ be an operator and put $S = U|T|^{1/2}$. Then $0 \in \sigma_r(S)$ if and only if $0 \in \sigma_r(T)$.*

Proof. Since $0 \in \sigma_r(S)$, we have $0 \in \sigma(S)$, $\ker S = \{0\}$ and there exist positive numbers c, d such that $||Sx|| \geq c||x||$ and $\||T|^{1/2}x\| \geq d\|x\|$ for $x \in \mathcal{H}$ (cf. [9, p. 69]). Hence, we have

$$||Tx|| = ||S|T|^{1/2}x|| \geq c|||T|^{1/2}x|| \geq cd||x||.$$

Since $0 \in \sigma(T)$ by Lemma 1 (3), we have $0 \in \sigma_r(T)$. Converse is similar, because $|T|$ is invertible. So the proof is complete.

We define a mapping $\tau(\cdot)$ on $\mathbf{C}$ by

$$\tau(re^{i\theta}) = \sqrt{r}\, e^{i\theta}.$$

Applying [10, Chapter I, Lemma 3.1] for $R = \mathbf{C} - \{0\}$ and the mapping τ, we have the following result.

Lemma 3. *Let $T = U|T|$ be a semi-hyponormal operator and put $S = U|T|^{1/2}$. Then*

(1) $\sigma_r(S) - \{0\} = \{\tau(z) : z \in \sigma_r(T) - \{0\}\}$,
(2) $\sigma(S) - \{0\} = \{\tau(z) : z \in \sigma(T) - \{0\}\}$.

By Lemmas 2 and 3, we have the following result.

Lemma 4. *Let $T = U|T|$ be a semi-hyponormal operator and put $S = U|T|^{1/2}$. Then*

(1) $\sigma_r(S) = \{\tau(z) : z \in \sigma_r(T)\}$,
(2) $\sigma(S) = \{\tau(z) : z \in \sigma(T)\}$.

Proof of Theorem 1. Put $S = U|T|^{1/2}$. Let $z = ae^{i\eta} \in \mathbf{C}$ belong to the residual spectrum $\sigma_r(T)$ of T. By Lemma 4 (2), it holds that $z \in \sigma_r(T)$ if and only if $w = \sqrt{a}\, e^{i\eta} \in \sigma_r(S)$. Since S is a pure hyponormal operator, by Theorem A, a point $w \in \mathbf{C}$ belongs to the residual spectrum of S if and only if there exists a compact ball B_1 centered at w such that

$$\iint_{\mathrm{B}_1} \frac{1 - g_S(u,v)}{|u+iv-w|^2}\, d\mu(u,v) \;<\; \infty.$$

By the transformation $x = \sqrt{a}\, u$ and $y = \sqrt{a}\, v$, we have

$$\iint_{\mathrm{B}_1} \frac{1 - g_S(u,v)}{|u+iv-w|^2}\, d\mu(u,v) \;=\; \iint_{\mathrm{B}} \frac{1 - g_S(a^{-1/2}\, x, a^{-1/2}\, y)}{|x+iy-z|^2}\, d\mu(x,y),$$

where B is some compact ball centered at z.

Here, since by the definition and Theorems D and E we have

$$g_S(x,y) = g_T(\sqrt{x^2+y^2} \cdot x, \sqrt{x^2+y^2} \cdot y),$$

it holds

$$\iint_{\mathrm{B}} \frac{1 - g_S(a^{-1/2}x, a^{-1/2}y)}{|x+iy-z|^2}\, d\mu(x,y)$$
$$= \iint_{\mathrm{B}} \frac{1 - g_T(a^{-1}\sqrt{x^2+y^2} \cdot x, a^{-1}\sqrt{x^2+y^2} \cdot y)}{|x+iy-z|^2}\, d\mu(x,y).$$

So the proof is complete.

Next we characterize the essential spectrum.

Theorem 2. *Let $T = U|T|$ be an invertible pure semi-hyponormal operator with $|T| - |T^*| = \xi \otimes \xi$. A point $z = ae^{i\eta} \in \mathbf{C}$ belongs to the essential spectrum of T if and only if the principal function is not almost everywhere equal to 0 or 1 in a neighbourhood of z.*

Proof. Let $S = U|T|^{1/2}$. Then S is an invertible pure hyponormal operator with $S^*S - SS^* = \xi \otimes \xi$. It is clear that $\sigma_{\text{ess}}(S) = \{\ \tau(z)\ :\ z \in \sigma_{\text{ess}}(T)\ \}$. Hence, by Theorem B it holds that $\tau(z) = \sqrt{a}\, e^{i\eta} \in \sigma_{\text{ess}}(S)$ if and only if the principal function $g_S(x, y)$ of S is not almost everywhere equal to 0 or 1 in a neighbourhood of $\sqrt{a}\, e^{i\eta}$. By the definition and Theorems D and E, we have

$$g_S(x, y) = g_S^P(e^{i\eta}, \sqrt{a}) = g_T^P(e^{i\eta}, a) = g_T(u, v),$$

where $x + iy = \sqrt{a}\, e^{i\eta}$ and $u + iv = a\, e^{i\eta}$ $(= z)$. It completes the proof.

3. The kernel function

Let $T = U|T|$ be a pure semi-hyponormal operator with $|T| - |T^*| = \xi \otimes \xi$. Put $S = U|T|^{1/2}$. For $z, w \notin \sigma(T)$, since $\tau(z), \tau(w) \notin \sigma(S)$, we define the kernel function $k_T(z, w)$ of T by

$$k_T(z, w) = \langle (S - \tau(w))^{*-1}\xi,\ (S - \tau(z))^{*-1}\xi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product. Hence, we have

$$k_T(z, w) = k_S(\tau(z),\ \tau(w)).$$

Theorem 3. *Let $T = U|T|$ be an invertible pure semi-hyponormal operator with $|T| - |T^*| = \xi \otimes \xi$. Then, for $z, w \notin \sigma(T)$,*

$$k_T(z, w) = 1 - \exp\left(-\frac{1}{\pi} \iint \frac{g_T(\sqrt{x^2 + y^2} \cdot x, \sqrt{x^2 + y^2} \cdot y)}{(x + iy - \tau(z))(x - iy - \overline{\tau(w)})}\, d\mu(x, y) \right). \tag{5}$$

Proof. Put $S = U|T|^{1/2}$. Then S is a pure hyponormal operator with

$$S^*S - SS^* = \xi \otimes \xi.$$

Hence, by [9, Chapter XI, Theorem 2.1] we have

$$k_S(\tau(z), \tau(w)) = 1 - \exp\left(-\frac{1}{\pi} \iint \frac{g_S(x, y)}{(x + iy - \tau(z))(x - iy - \overline{\tau(w)})}\, d\mu(x, y) \right).$$

Since $k_T(z, w) = k_S(\tau(z),\ \tau(w))$ and $g_S(x, y) = g_T(\sqrt{x^2 + y^2} \cdot x, \sqrt{x^2 + y^2} \cdot y)$, we have

$$k_T(z, w) = 1 - \exp\left(-\frac{1}{\pi} \iint \frac{g_T(\sqrt{x^2 + y^2} \cdot x, \sqrt{x^2 + y^2} \cdot y)}{(x + iy - \tau(z))(x - iy - \overline{\tau(w)})}\, d\mu(x, y) \right).$$

So the proof is complete.

By taking residues at infinity in (5), we have

$$\langle \xi,\ (S - \tau(z))^{*-1}\xi \rangle = \frac{1}{\pi} \iint \frac{g_T(\sqrt{x^2 + y^2} \cdot x, \sqrt{x^2 + y^2} \cdot y)}{x + iy - \tau(z)}\, d\mu(x, y). \tag{6}$$

$\mathcal{O}(A)$ denotes the set of all analytic functions on $A \subset \mathbf{C}$. Then we have the following result.

Theorem 4. *Let $T = U|T|$ be an invertible pure semi-hyponormal operator with $|T| - |T^*| = \xi \otimes \xi$. If $S = U|T|^{1/2}$, then, for $z \notin \sigma(T)$ and $f \in \mathcal{O}(\sigma(S))$,*

$$\langle f(S)\xi,\ (S-\tau(z))^{*-1}\xi\rangle = \frac{1}{\pi}\iint \frac{f(x+iy)g_T(\sqrt{x^2+y^2}\cdot x, \sqrt{x^2+y^2}\cdot y)}{x+iy-\tau(z)}\,d\mu(x,y).$$

Proof. First, for $\lambda \notin \sigma(T)$, since $\tau(\lambda) \notin \sigma(S)$, from the resolvent equality and (6) it follows that

$$\langle (S-\tau(\lambda))^{-1}\xi,\ (S-\tau(z))^{*-1}\xi\rangle = \langle \xi,\ (S-\tau(\lambda))^{*-1}(S-\tau(z))^{*-1}\xi\rangle$$
$$= \frac{1}{\pi}\iint \frac{g_T(\sqrt{x^2+y^2}\cdot x, \sqrt{x^2+y^2}\cdot y)}{(x+iy-\tau(\lambda))(x+iy-\tau(z))}\,d\mu(x,y).$$

Since

$$f(S) = \frac{1}{2\pi i}\int_\gamma f(w)(w-S)^{-1}\,dw,$$

it holds

$$\langle f(S)\xi,\ (S-\tau(z))^{*-1}\xi\rangle$$
$$= \frac{1}{2\pi i}\int_\gamma f(w)\langle (w-S)^{-1}\xi, (S-\tau(z))^{*-1}\xi\rangle dw$$
$$= \frac{1}{\pi}\iint \left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-(x+iy)}\,dw\right)\frac{g_T(\sqrt{x^2+y^2}\cdot x, \sqrt{x^2+y^2}\cdot y)}{x+iy-\tau(z)}d\mu(x,y)$$
$$= \frac{1}{\pi}\iint \frac{f(x+iy)g_T(\sqrt{x^2+y^2}\cdot x, \sqrt{x^2+y^2}\cdot y)}{x+iy-\tau(z)}\,d\mu(x,y).$$

So the proof is complete.

Remark. If an invertible operator $T = U|T|$ satisfies $|T|^{2p} - |T^*|^{2p} = \xi\otimes\xi$ for some p ($0 < p < 1/2$), then T is p-hyponormal and $|T|^{2p} - |T^*|^{2p} \in \mathcal{C}_1$. Put $S = U|T|^p$. Since $S^*S - SS^* = \xi \otimes \xi$, the operator S is an invertible hyponormal operator with one-dimensional self-commutator. Define a mapping $\tau_p(\cdot)$ by $\tau_p(re^{i\theta}) = r^p e^{i\theta}$. Then similar versions of Lemmas 1,2, 3 and 4 hold. Thus if an invertible operator $T = U|T|$ satisfies $|T|^{2p} - |T^*|^{2p} = \xi \otimes \xi$, then we have similar results.

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Spectral Consequences due to General Hölder's Inequalities and Domination of Semigroups

M. Demuth

Abstract. Let A be a generator of a strongly continuous positivity preserving semigroup in $L^r(\Omega)$ and V an admissible perturbation. Then Hölder's inequality

$$(e^{-t(A+M_V)}f)(x) \leq [(e^{-t(A+pM_V)}f)(x)]^{\frac{1}{p}}\ [(e^{-tA}f)(x)]^{\frac{1}{q}}$$

implies a kernel estimate

$$e^{-t(A+M_V)}(x,y) \leq c\,t^{-\alpha}\ e^{\omega t}(e^{-tA}(x,y))^{\frac{1}{q}}$$

with

$$\frac{1}{p}+\frac{1}{q}=1\,, \quad c\,,\alpha\,,\omega \quad \text{positive constants.}$$

The kernel estimate allows to study the spectra of $A+M_V$ if the spectrum of A is known. For instance we give conditions such that the essential and absolutely continuous spectra are stable. Moreover, one can prove the limiting absorption principle for $A+M_V$. The article is based on and continues the paper of W. Arendt, M. Demuth [2].

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1. Motivation

Let A be a generator of a strongly continuous, positivity preserving semigroup in $L^r(\Omega),\ 1 \leq r < \infty$, with respect to a $\sigma-$ finite measure space $(\Omega, \mathcal{F}, \mu)$. Let M_V be the multiplication operator with an admissible potential $V : \Omega \to \mathbb{R}_-$, such that $A+M_V$ generates a strongly continuous semigroup, too. In [2] we established a general pointwise Hölder's inequality for the semigroups of the form

$$\left(e^{-t(A+M_V)}f\right)(x) \leq \left[\left(e^{-t(A+pM_V)}f\right)(x)\right]^{\frac{1}{p}}\ \left[\left(e^{-tA}f\right)(x)\right]^{\frac{1}{q}} \tag{1.1}$$

where f is a non-negative function in $L^r(\Omega)$ and $1 < p,\ q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If the semigroups $\{e^{-tA},\ t \geq 0\},\ \{e^{-t(A+M_V)},\ t \geq 0\}$ consist of integral operators, (1.1) can be used to obtain a domination inequality of the form

$$\begin{aligned} e^{-t(A+M_V)}(x,y) &\leq \left[e^{-t(A+pM_V)}(x,y)\right]^{\frac{1}{p}} \left[e^{-tA}(x,y)\right]^{\frac{1}{q}} \\ &\leq c\, t^{-\alpha} e^{\omega t} \left[e^{-tA}(x,y)\right]^{\frac{1}{q}} \end{aligned} \tag{1.2}$$

for a.a. $x,\ y \in \Omega$ with $c,\ \alpha,\ \omega$ positive constants and $p,\ q > 1$.

The present article continues the results by Arendt, Demuth [2] in the sense that the domination estimate (1.2) is used to obtain several spectral properties of $A + M_V$. The simplest applications are Gaussian estimates. If the free kernel satisfies a Gaussian estimate, i.e., if

$$e^{-tA}(x,y) \ \leq\ c\, t^{-\alpha}\ e^{\omega t}\ e^{-\beta \frac{|x-y|^2}{t}} \tag{1.3}$$

$(c, \alpha, \omega, \beta$ non-negative constants) then an analogous estimate holds for $e^{-t(A+M_V)}$ (x, y). There is a huge literature on consequences of Gaussian estimates. See for instance Arendt [1], Arendt, ter Elst [3] Liskevich, Vogt, Voigt [10] or Ouhabaz [11]. If e^{-tB_r} satisfies a Gaussian estimate in $L^r(\Omega)$ for one fixed r then there are consistent semigroups $\{e^{-tB_s},\ t \geq 0\}$ on $L^s(\Omega)$ and $B_s = B_r$ for $1 \leq s < \infty$. If e^{-tB_r} is holomorphic then e^{-tB_s} is holomorphic and the spectra of B_r and B_s are equal, in particular the spectrum of B_r coincides with the spectrum of B_2.

However there are relevant generators A such that a Gaussian estimate does not hold. For instance, for fractional powers of the Laplacian, $A = (-\Delta)^\alpha,\ 0 < \alpha < 1$, in $L^2(\mathbb{R}^d)$, there is an estimate from below of the form

$$\frac{c}{t^{\frac{d}{2\alpha}}|x-y|^{d+2\alpha}} \ \leq\ e^{-tA}(x,y) \text{ for } \ |x-y| \geq 1 \text{ and } t \geq 1\ . \tag{1.4}$$

(1.4) prevents an estimate of the form (1.3).

Moreover, if A is a generator of a diffusion process, i.e., if A is associated to $\Sigma\ D_i\,(a_{ij}\,D_j)$ in $L^2(\mathbb{R}^d)$. Then the diffusion matrix $a_{ij}(x)$ can be reconstructed from the semigroup kernels by

$$a_{ij}(x) \ =\ \lim_{t\downarrow 0}\ \frac{1}{t} \int\limits_{|x-y|<1} (y_i - x_i)(y_j - x_j)\ e^{-tA}\ (x,y)\ dy\ .$$

If e^{-tA} satisfies a Gaussian estimate then $a_{ij}(x)$ are bounded necessarily. However, many diffusion processes have unbounded $a_{ij}(x)$. Thus Gaussian estimates of the form (1.3) are useful in several cases, but the free kernel estimate (1.2) is more general and includes further interesting situations.

That can be seen from the following considerations. Let A and $A + M_V$ be selfadjoint operators in $L^2(\mathbb{R}^d)$. Denote by

$$D_t \ = \ e^{-t(A+M_V)} - e^{-tA}, \ t > 0 \tag{1.5}$$

their semigroup difference. The essential spectra of A and $A + M_V$ coincide, if D_t is a Hilbert-Schmidt operator. The absolutely continuous spectra are equal if D_t is a trace-class operator. The Hilbert-Schmidt norm of D_t can be estimated by (see (4.4))

$$||D_t||_{\mathrm{HS}} \leq \left(2t \int\limits_{\mathbb{R}^d} e^{-2t(A+M_V)} \ (x,x) \ |V(x)| \ dx \right)^{\frac{1}{2}} \quad . \tag{1.6}$$

D_t is a trace class operator if

$$\int\limits_{\mathbb{R}^d} dx \ \sqrt{\int\limits_{\mathbb{R}^d} \left[e^{-\frac{t}{2}(A+M_V)} \ (x,y) \right]^2 \ |V(y)|^2 \ dy} \tag{1.7}$$

and

$$\int\limits_{\mathbb{R}^d} dx \ \sqrt{\int\limits_{\mathbb{R}^d} \left[e^{-\frac{t}{2}A} \ (x,y) \right]^2 \ |V(y)|^2 \ dy} \tag{1.8}$$

are finite.

Using the domination inequality (1.2) and the $L^1 - L^\infty$ smoothing property of e^{-tA} the conditions in (1.7) and (1.8) reduce to

$$\int\limits_{\mathbb{R}^d} dx \ \sqrt{\int\limits_{\mathbb{R}^d} e^{-\frac{t}{2}A} \ (x,y) \ |V(y)|^2 \ dy} < \infty \quad ,$$

see Proposition 4.2 .

For studying the limiting absorption principle (l.a.p.) one needs the finiteness of weighted L^2-norms of the resolvents. Let $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ be a weight function and assume

$$\left(e^{-tA} \varphi^2 \right)(x) \ \leq \ c_t \ \varphi^2(x) \quad ,$$

then Hölder's inequality (1.2) implies

$$\left(e^{-t(A+M_V)} \varphi \right)(x) \ \leq \ c_t \ \varphi(x) \quad .$$

This gives rise to the l.a.p. of $A + M_V$ if that for A is given (see (4.5)). Having the l.a.p one can study quantitatively for instance scattering matrices and scattering amplitudes. Further spectral applications follow from results in the book by Demuth, van Casteren [6].

2. Hölder's inequality for semigroups

Let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma-$finite measure space. Let A be the generator of a positivity preserving, strongly continuous semigroup $\{e^{-tA},\ t \geq 0\}$ on $L^r(\Omega),\ 1 \leq r < \infty$. Let $V : \Omega \to \mathbb{R}_-$ be a measurable function. Set $V_n(x) = \sup\{V(x),\ -n\}$, $n \in \mathbb{N}$. Let M_{V_n} be the multiplication operator with the bounded function V_n . Then $\operatorname{dom}(A + M_{V_n}) = \operatorname{dom}(A)$, $(A + M_{V_n})f = Af + M_{V_n} f$, and $A + M_{V_n}$ generates a strongly continuous semigroup $\{e^{-t(A+M_{V_n})},\ t \geq 0\}$ in $L^r(\Omega)$, too. The function V is called admissible if the limit $\lim\limits_{n\to\infty} e^{-t(A+M_{V_n})} f$ exists for any $t \geq 0$ and all $f \in L^r(\Omega)$ and is also a strongly continuous semigroup in $L^r(\Omega)$. Its generator is denoted by $A + M_V$. In the following we will allow only admissible V. For more details see Voigt [14], [15].

Proposition 2.1. *Let A and M_V be given as above. Let $1 < p,\ q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and assume pV to be admissible. Take nonnegative f in $L^r(\Omega)$. Then a pointwise Hölder's inequality holds, i.e.,*

$$\left[e^{-t(A+M_V)} f\right](x) \leq \left[\left(e^{-t(A+pM_V)} f\right)(x)\right]^{\frac{1}{p}} \left[\left(e^{-tA} f\right)(x)\right]^{\frac{1}{q}} \quad , \tag{2.1}$$

for a.a. $x \in \Omega$.

Proof. The proof is given by Arendt, Demuth in [2]. For the sake of completeness we repeat it here shortly. The proof is based on Trotter's product formula, which is true for bounded potentials V, i.e., for $V \in L^\infty(\Omega)$ we have

$$e^{-t(A+M_V)} = s - \lim_{n\to\infty} (e^{-\frac{t}{n}A}\ e^{-\frac{t}{n}M_V})^n \quad .$$

Let L^r_+ be the cone of nonnegative functions in $L^r(\Omega)$. $L^r_+ \cap \operatorname{dom}(A)$ is dense in L^r_+. Any $f \in L^r_+ \cap \operatorname{dom}(A)$ can be written as $f = uv$ with $v \in L^\infty(\Omega)$, $u \in \operatorname{dom}(A) \cap L^r_+$ and $u > 0$ a.e.

Therefore

$$(e^{-\frac{t}{n}A}\ e^{-\frac{t}{n}M_V})^n f \ = \ u \left(\frac{1}{u}\ e^{-\frac{t}{n}A}\ u\ e^{-\frac{t}{n}M_V}\right)^n v \quad .$$

Introducing a family of operators on $L^\infty(\Omega)$ by

$$S(t)g := \left(\frac{1}{u}\ e^{-tA}\ u\right) g \quad ,$$

$g \in L^\infty(\Omega)$, then $S(t)$ is a semigroup in $\mathfrak{B}(L^\infty)$. The pointwise Hölder's inequality in (2.1) finally follows from a corresponding inequality for $S(t)$, i.e., from

$$\left[\left(S\left(\frac{t}{n}\right)\ e^{-\frac{t}{n}M_V}\right)^n g\right](x) \tag{2.2}$$

$$\leq \left\{\left[\left(S\left(\frac{t}{n}\right)\ e^{-\frac{t}{n}pM_V}\right)^n g\right](x)\right\}^{\frac{1}{p}} \{(S(t)g)(x)\}^{\frac{1}{q}} .$$

For proving (2.2) one can use the Theorem of Gelfand-Naimark (see Rudin [12], 11.18). Let $C(.)$ be the set of continuous functions. For any Ω there is an algebra isomorphism J and a compact set K such that

$$J : C(K) \to L^\infty(\Omega)\,,$$

with $g(x) \geq 0$ iff $(Jg)(y) \geq 0$ $\mu-$almost everywhere. Setting

$$\begin{aligned} \tilde{S}(t) :&= J^{-1}S(t)J \quad , \quad C(K) \to C(K) \\ \tilde{g} :&= J^{-1}g \quad , \quad g \in L^\infty \quad , \\ \tilde{V} :&= J^{-1}V \quad , \quad V \in L^\infty \quad , \end{aligned}$$

then

$$\left(S\left(\frac{t}{n}\right) e^{-\frac{t}{n}M_V} \right)^n g = J\left(\tilde{S}\left(\frac{t}{n}\right) e^{-\frac{t}{n}M_{\tilde{V}}} J \right)^n \tilde{g} \quad .$$

Introducing the functional $\delta_x : C(K) \to \mathbb{R}$ by $\delta_x(\tilde{g}) := \tilde{g}(x)$ for $\tilde{g} \in C(K)$ we get

$$\left(\tilde{S}(t) * \delta_x\right)\tilde{g} = \left(\tilde{S}(t)\tilde{g}\right)(x).$$

On the other hand the theorem of Riesz yields the existence of a Radon measure $\mu_{t,x}(.)$ such that

$$\left(\tilde{S}(t) * \delta_x\right)(\tilde{g}) = \int_K \tilde{g}(y)\ \mu_{t,x}(dy) \quad ,$$

which implies

$$\left(\tilde{S}(t)\tilde{g}\right)(x) = \int_K \tilde{g}(y)\ \mu_{t,x}(dy) \quad .$$

Therefore,

$$\begin{aligned} \left[\left(\tilde{S}\left(\frac{t}{n}\right) e^{-\frac{t}{n}M_{\tilde{V}}} J \right)^n \tilde{g}\right](x) \quad &= \quad \int_K \cdots \int_K e^{-\frac{t}{n}\tilde{V}(y_n)} \ldots\ e^{-\frac{t}{n}\tilde{V}(y_1)}\ \tilde{g}(y_1) \\ & \qquad d\mu_{\frac{t}{n},\ y_2}(y_1) \cdots\ d\mu_{\frac{t}{n},\ x}(y_n) \quad . \end{aligned}$$

From the last integral (2.2) follows for $\tilde{S},\ \tilde{V},\ \tilde{g}$ by the usual Hölder's inequality. Substituting $S(t) = J\tilde{S}(t)J^{-1},\ V = J\tilde{V},\ g = J\tilde{g}$ we get (2.2) itself for any $g \in L^\infty(\Omega)$. In particular it holds for $v \in L^\infty(\Omega)$. For $f = uv$ thus we get

$$\begin{aligned} \left[\left(e^{-\frac{t}{n}A}e^{-\frac{t}{n}M_V}\right)^n f\right](x) \quad &\leq \quad \left\{\left[\left(e^{-\frac{t}{n}A}\ e^{-\frac{t}{n}pM_V}\right)^n f\right](x)\right\}^{\frac{1}{p}} \\ & \quad \cdot \left\{\left[e^{-tA}f\right](x)\right\}^{\frac{1}{q}} \quad . \end{aligned} \tag{2.3}$$

Then Trotter's product formula and the definition of admissible potentials complete the proof. □

Remark 2.2.

- Another proof of Proposition 2.1 is given by M. Haase in [8] avoiding the Gelfand-Naimark theorem.
- Note that we have not assumed an $L^1 - L^\infty$ smoothing of the semigroup e^{-tA}. This means that e^{-tA} may not be an integral operator.
- If A is the negative Laplacian then Kato class potentials are admissible (see, e.g., [13]).
- If A is a generator of a Feller process then Kato-Feller class potentials are admissible (see [6]).

3. Domination inequalities for the semigroup kernels

Now let $\{e^{-tA},\ t \geq 0\}$ be a positivity preserving, strongly continuous semigroup in $L^1(\Omega)$ which is ultracontractive of asymptotic dimension d, i.e., there is a $c > 0$ such that

$$||e^{-tA}||_{\mathfrak{B}(L^1,\ L^\infty)} \ \leq \ c\, t^{-\frac{d}{2}}, \quad 0 < t \leq 1 \quad . \tag{3.1}$$

Then e^{-tA} is an integral operator. Denoting its kernel by $e^{-tA}(x,y)$ we obtain for a.e. $x, y \in \Omega$

$$0 \leq e^{-tA}(x,y) \leq c\, t^{-\frac{d}{2}} \ ,$$

where $0 < t \leq 1$, or

$$0 \leq e^{-tA}(x,y) \leq c\ t^{-\frac{d}{2}}\ e^{\omega t} \tag{3.2}$$

for $t > 0$ and with positive constants $c,\ \omega$.

Let V, p, q be as in Proposition 2.1. From Hölder's inequality (Proposition 2.1)it follows that

$$\begin{aligned} &\left|\left|e^{-t(A+M_V)}\right|\right|_{\mathfrak{B}(L^1,L^p)} \\ &\leq c^{\frac{1}{q}}\ t^{-\left(\frac{d}{2}\cdot\frac{1}{q}\right)} \sup_{0<s\leq 1} \left|\left|e^{-s(A+pM_V)}\right|\right|^{\frac{1}{p}}_{\mathfrak{B}(L^1)} \end{aligned} \tag{3.3}$$

for $0 < t \leq 1$.

Proposition 3.1.

Let pV be an admissible potential for all $p > 1$ and assume additionally

$$\sup_{0<t\leq 1} \left|\left|\ e^{-t(A+pM_V)}\right|\right|_{\mathfrak{B}(L^\infty)} < \infty \quad . \tag{3.4}$$

Then the perturbed semigroup is ultracontractive of asymptotic dimension d.

Proof. This result is a consequence of the extrapolation theorem given by Arendt [1], 7.3.2. □

The last proposition implies that $e^{-t(A+M_V)}$ is an integral operator. We denote its kernel by $e^{-t(A+M_V)}(x,y)$. The kernel satisfies

$$0 \le e^{-t(A+M_V)}(x,y) \;\le\; c\, t^{-\frac{d}{2}}\, e^{\omega t} \tag{3.5}$$

for $t > 0$ and non-negative constants $c,\ \omega$.

Remark 3.2.

If $A = -\Delta$ then the assumption in Proposition 3.1 corresponds to Kashminskij's Lemma, which is satisfied for Kato-class potentials (see, e.g., [13]).

Having the semigroup kernels we will use the pointwise Hölder's inequality (2.1) for the domination of the kernels $e^{-t(A+M_V)}(x,y)$.

Very often Hölder's inequality is proved starting with the estimate

$$ab \;\le\; \frac{a^p}{p} \;+\; \frac{b^q}{q}$$

for $a,\ b > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. This can be sharpened to

$$ab = \inf_{s>0}\left(s^p\, \frac{a^p}{p} \;+\; \frac{1}{s^q}\, \frac{b^q}{q}\right) \tag{3.6}$$

Lemma 3.3. (*due to M. Haase* [9])

Let $(\Omega,\ \mathcal{F},\ \mu)$ *be a* $\sigma-$*finite measure space. Take* $1 \le p \le \infty$. *Let* $f_0,\ f_1$ *be positive functions such that*

$$\int_\Gamma f_1(x)\ \mu(dx) \;\le\; \mu(\Gamma)^{\frac{1}{q}} \left(\int_\Gamma f_0(x)^p\ \mu(dx)\right)^{\frac{1}{p}}$$

for all measurable $\Gamma \subset \Omega$ *with* $\mu(\Gamma) < \infty$.

Then

$$f_1(x) \;\le\; f_0(x) \quad \mu - a.e. \tag{3.7}$$

Proof. The cases $p = 1$ and $p = \infty$ are obvious. For $1 < p < \infty$ we get by (3.6)

$$\begin{aligned} ||f_1||_{L^1(\Gamma)} &\le ||f_0||_{L^p(\Gamma)}\ \mu(\Gamma)^{\frac{1}{q}} \\ &\le \frac{s^p}{p}\ ||f_0||^p_{L^p(\Gamma)} \;+\; \frac{1}{s^q}\ \frac{1}{q}\ \mu(\Gamma) \quad . \end{aligned}$$

Hence

$$f_1(x) \;\le\; \frac{s^p}{p}\ f_0^p(x) \;+\; \frac{1}{q}\ \frac{1}{s^q}$$

$\mu -$ a.e., $s > 0$. This is true for any s, i.e.,

$$
\begin{aligned}
f_1(x) &\le \inf_{s>0} \left(\frac{s^p}{p} \; f_0^p(x) \; + \; \frac{1}{q} \, \frac{1}{s^q} \right) \\
&= f_0(x) \cdot 1 \,,
\end{aligned}
$$

using (3.6) again. □

A similar argument is given by Liskevich, Vogt, Voigt, [10] in the proof of Lemma 3.9.

From Lemma 3.3 and Proposition 2.1 we get the following domination inequality.

Proposition 3.4.

Let $\{e^{-tA},\ t \ge 0\}$ be an ultracontractive strongly continuous semigroup. Let V be a potential satisfying the assumptions of Proposition 3.1.

Then the kernel of the perturbed semigroup can be estimated by

$$
e^{-t(A+M_V)}(x,y) \;\le\; \left(e^{-t(A+pM_V)}(x,y)\right)^{\frac{1}{p}} \cdot \; \left(e^{-tA}(x,y)\right)^{\frac{1}{q}} \tag{3.8}
$$

for a.e. $x, y \in \Omega$ *and with* $\frac{1}{p} + \frac{1}{q} = 1$.

More specifically

$$
e^{-t(A+M_V)}(x,y) \;\le\; c \;\cdot\; t^{-\frac{d}{2p}} \; e^{\frac{\omega}{p} t} \left(e^{-tA}(x,y)\right)^{\frac{1}{q}} \;. \tag{3.9}
$$

Proof. From Proposition 2.1 and from the proof of Lemma 3.3 we know that

$$
\begin{aligned}
&\int e^{-t(A+M_V)}(x,y) f(y) \; d\mu(y) \\
&\le \left(\int e^{-t(A+pM_V)}(x,y) \; f(y) \; d\mu(y)\right)^{\frac{1}{p}} \; \left(\int e^{-tA}(x,y) f(y) \; d\mu(y)\right)^{\frac{1}{q}} \\
&\le \frac{1}{p} \int e^{-t(A+pM_V)}(x,y) \; f(y) \; d\mu(y) \; + \; \frac{1}{q} \int e^{-tA}(x,y) \; f(y) \; d\mu(y) \,.
\end{aligned}
$$

Take a measurable $\Gamma \subset \Omega$ with $\mu(\Gamma) < \infty$ and $f(y) = \chi_\Gamma(y)$ then for a.a. $x, y \in \Omega$

$$
\begin{aligned}
&e^{-t(A+M_V)}(x,y) \\
&\le \frac{1}{p} \; e^{-t(A+pM_V)}(x,y) \; + \; \frac{1}{q} \; e^{-tA}(x,y) \,.
\end{aligned}
$$

Then Lemma 3.3 completes the proof of (3.8). The second inequality (3.9) follows from (3.5). □

The estimate in (3.8) allows to extend the pointwise Hölder's inequality to positive functions not in any $L^r(\Omega)$. Let f and g be positive functions, such that $(e^{-t(A+M_V)}fg)(x)$, $(e^{-t(A+pM_V)}f^p)(x)$ and $(e^{-tA}g^q)(x)$ exists almost everywhere, then (3.8) implies

$$\left(e^{-t(A+M_V)}fg\right)(x) \le \left[\left(e^{-t(A+pM_V)}f^p\right)(x)\right]^{\frac{1}{p}} \left[\left(e^{-tA}g^q\right)(x)\right]^{\frac{1}{q}} . \tag{3.10}$$

The terms in (3.10) are defined via the kernels of the corresponding semigroups.

In particular that estimate allows to study the semigroups in weighted L^r-spaces (see Section 4). Take $\Omega = \mathbb{R}^d$ and let $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ be a weight function, for instance $\varphi(x) = (1+|x|^2)^{\frac{s}{2}},\ s > 0$. Then the pointwise Hölder's inequality remains valid in the form

$$\left(e^{-t(A+M_V)}\varphi\right)(x) \ \le\ \left[\left(e^{-t(A+pM_V)}1\right)(x)\right]^{\frac{1}{p}}\ \left[\left(e^{-tA}\varphi^q\right)(x)\right]^{\frac{1}{q}} , \tag{3.11}$$

if the right-hand side exists for a.a. $x \in \mathbb{R}^d$. Hence we can formulate the following corollary.

Corollary 3.5.

Let pV be admissible and assume $e^{-t(A+pM_V)} \in \mathfrak{B}(L^\infty)$. Let φ be a positive function such that $(e^{-tA}\varphi^q)(x)$ is finite for a.e. $x \in \mathbb{R}^d$. Then

$$\left(e^{-t(A+M_V)}\varphi\right)(x) \ \le\ \left|\left|e^{-t(A+pM_V)}\right|\right|^{\frac{1}{p}}_{\mathfrak{B}(L^\infty)}\ \left\{\left[e^{-tA}\varphi^q\right](x)\right\}^{\frac{1}{q}} .$$

4. Spectral consequences

From the kernel estimates in Proposition 3.4 it is obvious that Gaussian estimates remain true for $e^{-t(A+M_V)}$ if they hold for the free semigroup. As mentioned in Section 1 the literature about consequences of Gaussian estimates is huge (see, e.g., Arendt [1]; Davies[4]; Liskevich et al. [10]; Ouhabaz [11]). Here, we will concentrate on consequences of the domination inequality in Proposition 3.4 without assuming a Gaussian estimate for e^{-tA}.

At first we note the following. The kernel estimate (3.8) implies

$$\begin{aligned} ||e^{-t(A+M_V)}||_{\mathfrak{B}(L^1,\ L^\infty)} &\le\ c\, e^{\frac{\omega t}{p}}\ t^{-\frac{d}{2p}}\ ||e^{-tA}||^{\frac{1}{q}}_{\mathfrak{B}(L^1,\ L^\infty)} \\ &\le\ c\, e^{\frac{\omega t}{p}}\ t^{-\frac{d}{2}\ (\frac{1}{p}+\frac{1}{q})} =\ c\, e^{\frac{\omega t}{p}}\ t^{-\frac{d}{2}} . \end{aligned} \tag{4.1}$$

Thus, if one has to estimate powers of resolvents the singularity at $t = 0$ is not changed by the perturbation. For instance

$$||(A+a)^{-r} -\ (A+M_V+a)^{-r}||_{\mathfrak{B}(L^1,\ L^\infty)} \le\ c \int_0^\infty \lambda^{r-1-\frac{d}{2}}\ e^{-a\lambda}\ e^{\omega\lambda}\ d\lambda . \tag{4.2}$$

That is useful for many spectral theoretical consequences where resolvent differences play a major role.

In the following we consider selfadjoint generators A and $A + M_V$ in $L^2(\Omega)$. For Ω we can take a second countable locally compact Hausdorff space with a Borel field $\mathcal{F}$ and a Radon measure μ. On the other hand in many applications we restrict us to operators in $L^2(\mathbb{R}^d)$. Take again only $V : \Omega \to \mathbb{R}_-$. Assume that e^{-tA}, $e^{-t(A+M_V)}$ are integral operators for any $t > 0$ and that they are positivity preserving. Then the semigroup kernels are symmetric, i.e.,

$$\begin{aligned} e^{-tA}(x,y) &= e^{-tA}(y,x) \quad , \\ e^{-t(A+M_V)}(x,y) &= e^{-t(A+M_V)}(y,x) \end{aligned}$$

and

$$e^{-tA}(x,y) \le e^{-t(A+M_V)}(x,y)$$

for almost all $x, y \in \Omega$. Moreover we assume that all the kernels are continuous in x and y.

Define $D_t : = e^{-t(A+M_V)} - e^{-tA}$, the semigroup difference. D_t is also an integral operator. In [5] the so called comparison function,

$$D_t(x) = \int_\Omega | \, e^{-t(A+M_V)}(x,y) - e^{-tA}(x,y) \, | \, d\mu(y) \, , \tag{4.3}$$

was defined and studied. If all the semigroups are $L^1 - L^\infty$ and $L^\infty - L^\infty$ smoothing then the following spectral results are true for any fixed $t > 0$ (see [5] p. 103).

a) D_{2t} is a trace class operator, if $\sqrt{D_t(\cdot)} \in L^1$. The trace norm can be estimated in terms of $\int_\Omega \sqrt{D_t(x)} \, d\mu(x)$.

b) D_{2t} is a Hilbert-Schmidt operator if $D_t(.) \in L^2$, and

$$||D_t||^2_{\text{HS}} \le c \int_\Omega |D_t(x)|^2 \, d\mu(x).$$

c) The absolutely continuous spectra of A and $A+M_V$ coincide if $D_t(.) \in L^1(\Omega)$.

In all these results the domination inequalities (3.8) and (3.9) can be used for obtaining conditions on V. In the following we assume that pV is admissible for all $p > 1$ such that (3.8) holds.

Here we will summarize some of the applications.

Proposition 4.1. *D_t is a Hilbert-Schmidt operator if*

$$\int_\Omega e^{-2t(A+M_V)}(x,x) \, |V(x)| \, d\mu(x) < \infty \, . \tag{4.4}$$

This is in particular true if

$$V \in L^1(\Omega) \text{ and } e^{-tA} \text{ is } L^1 - L^\infty \text{ smoothing.}$$

Proof. The proof is taken from [6] Theorem 5.7. For the consequences one can use (3.8).

$$\begin{aligned}
||D_t||^2_{\mathrm{HS}} &= \int_\Omega \int_\Omega \left(e^{-t(A+M_V)}(x,y) - e^{-tA}(x,y)\right)^2 \; d\mu(x)\; d\mu(y) \\
&\le \int_\Omega \int_\Omega \left\{\left(e^{-t(A+M_V)}(x,y)^2 - e^{-tA}(x,y)^2\right)\right\} \; d\mu(x)\; d\mu(y) \\
&= \int_\Omega \left\{e^{-2t(A+M_V)}(x,x) - e^{-2tA}(x,x)\right\} \; d\mu(x) \\
&\quad \text{(symmetry and semigroup property)} \\
&\le \int_0^{2t} \int_\Omega \int_\Omega e^{-(2t-s)(A+M_V)}(x,u)\; |V(u)|\; e^{-sA}(u,x)\; d\mu(x)\; d\mu(u)\; ds \\
&\le c \int_0^{2t} \int_\Omega e^{-2t(A+M_V)}(u,u)\; |V(u)|\; d\mu(u)\; ds \\
&= c \cdot 2t \int_\Omega e^{-2t(A+M_V)}(u,u)\; |V(u)|\; d\mu(u)\,. \qquad \square
\end{aligned}$$

In a similar way one can show that D_t is a Hilbert-Schmidt operator if

$$\int_\Omega e^{-2t(A+M_V)}(x,x)\; |V(x)|^2\; d\mu(x) < \infty\,.$$

Therefore $V \in L^2$ is sufficient if the free semigroup is $L^1 - L^\infty$ smoothing and because of (3.9), which is assumed to hold.

Proposition 4.2.

D_t is a trace class operator if

$$\int_\Omega \sqrt{\int_\Omega e^{-\frac{t}{2}A}(x,y)\,|V(y)|^2\; d\mu(y)} \quad d\mu(x)$$

is finite.

Remark: Then it is known that the absolutely continuous spectra of A and $A+M_V$ coincide.

Proof. Following Theorem 6.7 in [6] D_t is trace class if

$$\int_\Omega \sqrt{\int_\Omega \left(e^{-\frac{t}{2}(A+M_V)}(x,y)\right)^2 \; |V(y)|^2\; d\mu(y)} \quad d\mu(x)$$

together with

$$\int_\Omega \sqrt{\int_\Omega \left(e^{-\frac{t}{2}A}(x,y)\right)^2 \; |V(y)|^2 \; d\mu(y)} \quad d\mu(x)$$

are finite. Using (3.9) for $q = 2$ and $\sup\limits_{x,y} \; e^{-\frac{t}{2}A}(x,y) < \infty$ ($L^1 - L^\infty$ smoothing) the condition in the proposition is sufficient. □

The stability of the absolutely continuous spectrum is studied also in the next proposition.

Proposition 4.3.

The absolutely continuous spectra of A and $A + M_V$ coincide if $V \in L^1(\Omega)$, if $e^{-tA} \in \mathfrak{B}(L^1, \; L^\infty)$ and if

$$\operatorname*{ess\,sup}_{u\in\Omega} \int_0^t \; ds \; s^{-\frac{d}{4}} \int_\Omega d\mu(x) \;\; \left[e^{-sA}(x,u)\right]^{\frac{1}{2}} < \infty \quad .$$

Proof. For the stability of the absolutely continuous spectra it is sufficient that $e^{-tA} \; D_t \; e^{-t(A+M_V)}$ is trace class. This is true if

$$\int_\Omega D_t(x) \; dx < \infty.$$

With Duhamel's formula the integral above is smaller or equal to

$$\begin{aligned}
&\int_\Omega d\mu(x) \int_\Omega d\mu(y) \int_\Omega d\mu(u) \int_0^t \; ds \\
&e^{-s(A+M_V)}(x,u) \; |V(u)| \; e^{-(t-s)A}(u,y) \\
&\leq \; c \int_0^t ds \int_\Omega d\mu(x) \int_\Omega d\mu(u) \; e^{-s(A+M_V)}(x,u) \; |V(u)| \\
&\leq ||V||_{L^1} \; \sup_u \int_0^t ds \int_\Omega d\mu(x) \; e^{-s(A+M_V)}(x,u) \, .
\end{aligned}$$

Then one can apply (3.9) for instance for $p = q = 2$. □

For a selfadjoint operator the limiting absorption principle (l.a.p.) is one possibility to decide whether the spectrum of A is locally purely absolutely continuous. Resolvent matrix elements for resolvent values approaching the spectrum determine the spectral family $E_A(.)$ of A . Take $\Delta = (\lambda_1,\ \lambda_2),\ 0 < \lambda_1 < \lambda_2$, assume that λ_1 and λ_2 are no eigenvalues of A, then the spectral measure of A on Δ is given by

$$\langle E_A(\Delta)f,\ g\rangle = \lim_{\epsilon\downarrow 0} \frac{1}{2\pi i} \int_{\lambda_1}^{\lambda_2} < [(A-\lambda-i\epsilon)^{-1} - (A-\lambda+i\epsilon)^{-1}]\ f,\ g > d\lambda\ ,$$

for appropriate $f,\ g \in L^2$.

Hence one is interested in weighted L^2-norms of the resolvents. In the following we assume $\Omega = \mathbb{R}^d$ and that

$$\sup_{\lambda\in\Delta}\ \sup_{\epsilon\in(0,1)} ||\varphi^{-1}(A-\lambda\pm i\epsilon)^{-1}\varphi^{-1}||_{\mathfrak{B}(L^2)} =: d_\Delta \tag{4.5}$$

is finite. Here φ are certain weight functions. We can have in mind for instance $\varphi(x) = (1+|x|^2)^{\frac{s}{2}},\ s > 0$. Our aim is to verify the l.a.p. for the perturbed operator $A + M_V$. Due to the inequality

$$\begin{aligned} &||\varphi^{-1}(A+M_V-\lambda+i\epsilon)^{-1}\ \varphi^{-1}||_{\mathfrak{B}(L^2)} \\ &\le d_\Delta + ||\varphi^{-1}[(A+M_V-\lambda+i\epsilon)^{-1} - (A-\lambda+i\epsilon)^{-1}]\varphi^{-1}||_{\mathfrak{B}(L^2)} \end{aligned} \tag{4.6}$$

it suffices to consider the last difference. Here the Hilbert identity is useful:

$$\begin{aligned} &(A+M_V-\lambda+i\epsilon)^{-1} - (A-\lambda+i\epsilon)^{-1} \\ &= [1+(\lambda-i\epsilon+a)(A+M_V-\lambda+i\epsilon)^{-1}] \\ &\quad [(A+M_V+a)^{-1} - (A+a)^{-1}] \\ &\quad [1+(\lambda-i\epsilon+a)(A-\lambda+i\epsilon)^{-1}], \end{aligned} \tag{4.7}$$

where $a > 0$ and $-a$ is any real resolvent value of both A and $A + M_V$.

In (4.7) we multiply both sides with φ^{-1} and introduce $\varphi\varphi^{-1}$ inside the formula twice. Then

$$\begin{aligned} &||\varphi^{-1}(A+M_V-\lambda+i\epsilon)^{-1}\varphi^{-1}||_{\mathfrak{B}(L^2)} \\ &\le\ d_\Delta + [1+(\lambda_2+a+1)\ d_\Delta] \\ &\quad \cdot\ \big([1+(\lambda_2+a+1)\ ||\varphi^{-1}(A+M_V-\lambda+i\epsilon)^{-1}\varphi^{-1}||_{\mathfrak{B}(L^2)}\big) \\ &\quad ||\varphi[(A+M_V+a)^{-1} - (A+a)^{-1}]\varphi||_{\mathfrak{B}(L^2)}. \end{aligned} \tag{4.8}$$

Thus we obtain the limiting absorption principle for $A + M_V$ and $\lambda \in \Delta$ if the norm of $\varphi[(A+M_V+a)^{-1} - (A+a)^{-1}]\varphi$ is small enough.

Our objective here is not to find the most general or best estimate. We will only show that the general Hölder's inequalities (3.10) and (3.11) are very useful to clarify the problem above.

Lemma 4.4. *Assume A, M_V as described above. Let $e^{-t(A+2M_V)}$ be $L^\infty - L^\infty$ smoothing. Let $f \in L^2(\mathbb{R}^d)$ and φ be a weight function.*

Then

$$\begin{aligned}&|((A+M_V+a)^{-1}\ \varphi f)(x)|^2\\&\quad\leq\ |((A+2M_V+a)^{-1}1)(x)|\ |((A+a)^{-1}\ \varphi^2\ |f|^2)(x)|\\&\quad\leq\ c\ |((A+a)^{-1}\ \varphi^2\ |f|^2)(x)|\,,\end{aligned}$$

where $a>0$ is assumed to be large enough.

Proof. From the Hölder's inequality (3.10) we have

$$\begin{aligned}&|[(A+M_V+a)^{-1}\ \varphi\ f](x)|^2\\&\quad=\ \left|\int_0^\infty d\lambda\ e^{-a\lambda}(e^{-\lambda(A+M_V)}\ \varphi f)(x)\right|^2\\&\quad\leq\ \left\{\int_0^\infty d\lambda\ e^{-a\lambda}(e^{-\lambda(A+2M_V)}1)(x)^{\frac12}\ (e^{-\lambda A}\varphi^2|f|^2)(x)^{\frac12}\right\}^2\\&\quad\leq\ \int_0^\infty d\lambda\ e^{-a\lambda}(e^{-\lambda(A+2M_V)}1)(x)\ \cdot\ \int_0^\infty d\lambda\ e^{-a\lambda}(e^{-\lambda A}\ \varphi^2|f|^2)(x)\\&\quad\leq\ c\left(\int_0^\infty d\lambda\ e^{-a\lambda}\ e^{\omega\lambda}\right)((A+a)^{-1}\varphi^2|f|^2)(x)\,.\end{aligned}$$

□

Proposition 4.5. *Under the same assumptions as above*

$$||\ \varphi\ [(A+a)^{-1}-(A+M_V+a)^{-1}]\ \varphi\ ||\quad\leq\quad c\ ||\ \varphi^2(A+a)^{-1}|V|\ ||_{L^\infty}\,.$$

Proof. We consider

$$\begin{aligned}&|\left\langle g,\ \varphi[(A+a)^{-1}-(A+M_V+a)^{-1}]\ \varphi\ f\right\rangle|\\&\quad\leq\ \left\langle |V|^{\frac12}\ (A+a)^{-1}\varphi\ |g|\ ,\ |V|^{\frac12}(A+M_V+a)^{-1}\ \varphi\ |f|\ )\right\rangle\\&\quad\leq\ \int_{\mathbb{R}^d}dx\int_{\mathbb{R}^d}dy\ |V(x)|^{\frac12}\ (A+a)^{-1}(x,y)\ \varphi(y)\ |g(y)|\\&\qquad\qquad|V(x)|^{\frac12}[(A+M_V+a)^{-1}\ \varphi\ |f|\ ](x)\\&\quad\leq\ \left(\int_{\mathbb{R}^d}dx\int_{\mathbb{R}^d}dy\ |V(x)|\ (A+a)^{-1}(x,y)\ \varphi^2(y)\ |g(y)|^2\right)^{\frac12}\\&\qquad\left(\int_{\mathbb{R}^d}dx\int_{\mathbb{R}^d}dy\ |V(x)|\ (A+a)^{-1}(x,y)\cdot|[(A+M_V+a)^{-1}\ \varphi\ |f|\ ](x)|^2\right)^{\frac12}\end{aligned}$$

$$\leq \quad c \,||\, |V|(A+a)^{-1} \; \varphi^2 \; |g|^2 \; ||_{L^1}^{\frac{1}{2}}$$

$$\left(\int_{\mathbb{R}^d} dx \; |V(x)| \; |[(A+M_V+a)^{-1} \; \varphi \; |f| \;](x)|^2 \right)^{\frac{1}{2}} .$$

By Lemma 4.4 and due to Hölder's inequality (3.11) the last expression is smaller than

$$\begin{aligned} c \,||\, |V|(A+a)^{-1} \; \varphi^2 \; |g|^2 \; ||_{L^1}^{\frac{1}{2}} \quad & ||\, |V|(A+a)^{-1} \; \varphi^2 \; |f|^2 \; ||_{L^1}^{\frac{1}{2}} \\ \leq \quad & c \,\|\, |V|(A+a)^{-1} \; \varphi^2||_{\mathfrak{B}(L^1)} \; ||f||_{L^2} \; ||g||_{L^2} \\ = \quad & c \; ||\varphi^2(A+a)^{-1} \; |V| \; ||_{L^\infty} \; ||f||_{L^2} \; ||g||_{L^2} \, . \end{aligned}$$

□

Corollary 4.6. *Let $\Delta = (\lambda_1, \; \lambda_2)$. Assume the l.a.p. for the free operator A with respect to Δ. Set $\lambda_2 + a + 1 = \alpha$.*

Assume V to fulfill

$$||\varphi^2(A+a)^{-1}|V| \; ||_{L^\infty} \leq \frac{1}{2\alpha} \; \frac{1}{1+\alpha d_\Delta} \quad . \tag{4.9}$$

Then the limiting absorption principle for $A + M_V$ holds and

$$||\varphi^{-1}(A+M_V-\lambda+i\epsilon)^{-1} \; \varphi^{-1}||_{\mathfrak{B}(L^2)} \leq 2 \; \left(d_\Delta + \frac{1}{2\alpha} \right) \quad .$$

Proof. Set $||\varphi^2(A+a)^{-1}|V| \; ||_{L^\infty} = r_V$ and $||\varphi^{-1}(A+M_V-\lambda+i\epsilon)^{-1} \; \varphi^{-1}||_{\mathfrak{B}(L^2)} = d_V$.

Then (4.8) reads

$$d_V \; \leq \; d_\Delta + (1+\alpha \; d_\Delta) \; (1+\alpha \; d_V) \; r_V \, ,$$

or

$$d_V \; (1-\alpha(1+\alpha \; d_\Delta) \; r_V) \leq \; d_\Delta + (1+\alpha \; d_\Delta) \; r_V \, .$$

With (4.9) it follows

$$d_V \leq 2 \; \left(d_\Delta \; + \; \frac{1}{2\alpha} \right) .$$

□

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Quadratically Hyponormal Recursively Generated Weighted Shifts

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Abstract. Given $0 < a < b < 1$, choose $c > b$ so that the recursively generated weight sequence $(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ has norm 1. For n variables $0 < x_1 < \cdots < x_n < a$, let $\widehat{\alpha}(x_1, \ldots, x_n) : \sqrt{x_n}, \ldots, \sqrt{x_1}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ and let $W_{\widehat{\alpha}(x_1,\ldots,x_n)}$ be a weighted shift with weight sequence $\widehat{\alpha}(x_1, \ldots, x_n)$. It is known that in the case of $n = 1$, positive quadratic hyponormality of $W_{\widehat{\alpha}(x)}$ is equivalent to quadratic hyponormality. But it has not been known whether these two notions are equivalent for $n \geq 2$. In this note, we obtain a useful characterization for positive quadratic hyponormality of $W_{\widehat{\alpha}(x_1,\ldots,x_n)}$ which, when specialized to $n = 2$, settles (in the negative) a conjecture that positive quadratic hyponormality of $W_{\widehat{\alpha}(x_1,x_2)}$ is equivalent to its quadratic hyponormality.

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1. Introduction

Let $\mathcal{H}$ be a separable, infinite-dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $A, B \in \mathcal{L}(\mathcal{H})$ let $[A, B] := AB - BA$. We say that an n-tuple $T = (T_1, \ldots, T_n)$ of operators in $\mathcal{L}(\mathcal{H})$ is *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. For $n \geq 1$ and $T \in \mathcal{L}(\mathcal{H})$, T is *n-hyponormal* if $(I, T, \ldots, T^n)$ is hyponormal ([3]). Recall that $T = (T_1, \ldots, T_n)$ is *weakly hyponormal* if $\lambda_1 T_1 + \cdots + \lambda_n T_n$ is hyponormal for every $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, n$, where $\mathbb{C}$ is the set of complex numbers. An operator T is *weakly n-hyponormal* if $(T, T^2, \ldots, T^n)$ is weakly hyponormal ([3],[9]). In particular, weak 2-hyponormality is often referred to as *quadratic hyponormality*, which is equivalent to $T + sT^2$ hyponormal for every $s \in \mathbb{C}$. A related definition is that of positive quadratic hyponormality (introduced in [3], [4] and which will be recalled below). Weighted shifts of two types have been studied to detect differences in these classes, namely Bergman type and recursive type.

If $W_{\alpha(x)}$ is a weighted shift of Bergman type with a weight sequence $\alpha(x)$: $\sqrt{x}, \sqrt{2/3}, \sqrt{3/4}, \sqrt{4/5}, \ldots$ and $x > 0$, then its quadratic hyponormality is equivalent to positive quadratic hyponormality; more precisely, the sequence $\alpha(x)$: $\sqrt{x}, \sqrt{2/3}, \sqrt{3/4}, \sqrt{4/5}, \ldots$ induces the same interval $(0, 2/3]$ in x for positive quadratic hyponormality and quadratic hyponormality for the weighted shift $W_{\alpha(x)}$ ([3],[4]). But for $\alpha(x,y) : \sqrt{x}, \sqrt{y}, \sqrt{3/4}, \sqrt{4/5}, \ldots$, the shift $W_{\alpha(5/8,5/8)}$ is quadratically hyponormal but not positively quadratically hyponormal ([10]). As well, for a weight sequence of recursive type $\widehat{\alpha}(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^{\wedge}$ with $0 < x < a < b < c$ (definition to be reviewed below), Curto-Fialkow obtained the value

$$h_2^+ := (\sup\{x : W_{\widehat{\alpha}(x)} \text{ is positively quadratically hyponormal}\})^{1/2}$$

in terms of a, b, and c. They asked in [6] whether positive quadratic hyponormality of $W_{\widehat{\alpha}(x)}$ is equivalent to quadratic hyponormality. This was solved affirmatively in [10], and so the following question naturally arises.

Problem 1.1. Let $\widehat{\alpha}\,(x_1, \ldots, x_k)$: $\sqrt{x_k}, \ldots, \sqrt{x_1}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^{\wedge}$ with $0 < x_k \leq \cdots \leq x_1 \leq a \leq b \leq c$. Is it true for $k \geq 2$ that positive quadratic hyponormality of $W_{\widehat{\alpha}(x_1,\ldots,x_k)}$ is equivalent to quadratic hyponormality?

This note compares positive quadratic hyponormality and quadratic hyponormality for weighted shifts of recursive type and consists of four sections as follows. In Section 2 we recall some notation and terminology from [8] for the reader's convenience. In Section 3 we characterize positive quadratic hyponormality for $W_{\widehat{\alpha}(x_1,x_2)}$. In Section 4 we characterize quadratic hyponormality for the general back step extension $W_{\widehat{\alpha}(x_1,\ldots,x_k)}$, which also yields quadratic hyponormality for the 2 step extension $W_{\widehat{\alpha}(x_1,x_2)}$; combining the results gives a negative answer for Problem 1.1.

2. Preliminaries

Let $\{e_i\}_{i=0}^{\infty}$ be the canonical orthonormal basis for $l^2(\mathbb{Z}_+)$, and let $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ be a bounded sequence of positive numbers. Let W_α be the associated unilateral weighted shift defined on $l^2(\mathbb{Z}_+)$. For $s \in \mathbb{C}$, we let

$$\begin{aligned} D_n(s) &= P_n\left[\left(W_\alpha + sW_\alpha^2\right)^*, \left(W_\alpha + sW_\alpha^2\right)\right] P_n \\ &= \begin{pmatrix} q_0 & \overline{r}_0 & 0 & \cdots & 0 & 0 \\ r_0 & q_1 & \overline{r}_1 & \cdots & 0 & 0 \\ 0 & r_1 & q_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_{n-1} & \overline{r}_{n-1} \\ 0 & 0 & 0 & \cdots & r_{n-1} & q_n \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} q_k &:= u_k + |s|^2 v_k, \\ r_k &:= s\sqrt{w_k}, \\ u_k &:= \alpha_k^2 - \alpha_{k-1}^2, \\ v_k &:= \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2, \\ w_k &:= \alpha_k^2\left(\alpha_{k+1}^2 - \alpha_{k-1}^2\right)^2 \quad (k \geq 0), \end{aligned}$$

and $\alpha_{-1} = \alpha_{-2} := 0$, and P_n denotes the orthogonal projection onto the subspace generated by $e_0, \ldots, e_n$. Since W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for all $s \in \mathbb{C}$ and all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, one considers $d_n(\cdot) := \det(D_n(\cdot))$. Note that d_n is a polynomial in $t := |s|^2$ of degree $n+1$, with Maclaurin expansion

$$d_n(t) := \sum_{i=0}^{n+1} c(n,i)\,t^i.$$

We say that W_α is *positively quadratically hyponormal* if $c(n,i) \geq 0$ for all $n, i \geq 0$ with $0 \leq i \leq n+1$. By direct computations we have

$$d_0 = q_0, \qquad d_1 = q_0 q_1 - |r_0|^2,$$

$$d_{n+2} = q_{n+2} d_{n+1} - |r_{n+1}|^2 d_n \quad (n \geq 0).$$

Recall also that given weights $\alpha : \alpha_0, \alpha_1, \alpha_2$ with $0 < \alpha_0 < \alpha_1 < \alpha_2$, there is a canonical way to generate a weight sequence recursively due to Stampli [11]. Define

$$\widehat{\alpha}_n = \left(\Psi_1 + \frac{\Psi_0}{\alpha_{n-1}^2}\right)^{1/2} \qquad (n \geq 3),$$

where

$$\Psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2}, \qquad \Psi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}.$$

This yields a bounded sequence $\widehat{\alpha} := \{\widehat{\alpha}_i\}_{i=0}^{\infty}$, where $\widehat{\alpha}_i = \alpha_i$ $(0 \leq i \leq 2)$ and gives also the weighted shift operator $W_{\widehat{\alpha}}$ (or written $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$) ([5],[6],[7],[11]). In general, if W_α is a weighted shift, then 2-hyponormality of W_α implies positive quadratic hyponormality and also 2-hyponormality of W_α is equivalent to $u_2 v_3 - w_2 \geq 0$ ([1],[3]).

Some of the calculations in this article were obtained through computer experiments using the software tool Mathematica [12].

3. Positive quadratic hyponormality

Let $\widehat{\alpha}(x,y) : \sqrt{y}, \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ with $0 < y \leq x < a < b < c$ be a weight sequence. (Recall from [2] that for any quadratically hyponormal weighted shift W_α with weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$, if $\alpha_k = \alpha_{k+1}$ for $k \geq 1$, then $\alpha_k = \alpha_{k+1}$ for all $k \geq 1$. Thus it is sufficient to study parameters with the strict inequalities above.)

First, we will consider (as in the last part of [8]) the weighted shift corresponding to $(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge$ with c chosen so the shift has norm exactly one (and with $0<a<b<1$, of course). This is without loss of generality for matters of positive quadratic hyponormality and quadratic hyponormality because both are preserved under scaling. The relevant choice of c is

$$c=c_{\text{norm}}:=\frac{a(b^2-b+1)-b}{(a-1)b}.$$

For the weight sequence $\widehat{\alpha}\,(x,y)$, we can obtain directly

$$\begin{aligned}
c\,(0,0) &= y,\ c\,(0,1)=xy,\\
c\,(1,0) &= (x-y)y,\ c\,(1,1)=-yx\,(y-a)\,,\ c\,(1,2)=ax^2y,\\
c\,(2,0) &= y\,(a-x)\,(x-y)\,,\ c\,(2,1)=ya\,(y-x)\,(x-b)\,,\\
c\,(2,2) &= b(a-y)-x(x-y)\geq (b-x)\,(a-y),\\
c\,(2,3) &= (ab-xy)\,yx^2a.
\end{aligned}$$

Thus $c(n,i)\geq 0$ $(0\leq n\leq 2, 0\leq i\leq n+1)$. By standard facts ([1]),

$$\begin{aligned}
c(n,n+1) &= v_0\cdots v_n\geq 0\quad (n\in\mathbb{N}_0),\\
c(n,0) &= u_0\cdots u_n\geq 0\quad (n\in\mathbb{N}_0).
\end{aligned}$$

And we may check individually (that is, include in our conditions) positivity of

$$\begin{aligned}
&c(3,1),c(3,2),c(3,3),\\
&c(4,2),c(4,3),\\
&c(5,3),\quad \text{and}\\
&c(6,4)\quad \text{(which will be redundant in some cases).}
\end{aligned}$$

An argument as in [8] yields the following: for $n\geq 4$,

$$c(n,i)=\begin{cases}
v_n\cdots v_3c(2,3)=v_n\cdots v_0 & (i=n+1),\\
u_nc(n-1,n)+v_n\cdots v_4\delta(2) & (i=n),\\
u_nc(n-1,n-1)+v_n\cdots v_4\delta(1) & (i=n-1),\\
u_nc(n-1,n-2)+v_n\cdots v_4\delta(0) & (i=n-2),\\
u_nc(n-1,i) & (0\leq i\leq n-3),
\end{cases}\tag{1}$$

where

$$\begin{aligned}
\delta(2) &:= v_3c(2,2)-w_2c(1,2),\\
\delta(1) &:= v_3c(2,1)-w_2c(1,1),\\
\delta(0) &:= v_3c(2,0)-w_2c(1,0).
\end{aligned}$$

Some easy computations show that

$$\delta(0)\geq 0\iff u_2v_3-w_2\geq 0\iff W_{\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}\text{ is 2-hyponormal}$$

for $0<x\leq a\leq b\leq c$. This 2-hyponormality condition should be compared to positive quadratic hyponormality of $W_{\sqrt{y},\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ as characterized in Theorem 3.2.

Suppose for a moment that we have ensured

$$c(n, n-i) \geq 0, \qquad n \geq 0,\ 0 \leq i \leq 2. \tag{2}$$

Repeated use of the last line of (1) then shows $c(n,i) \geq 0$ $(0 \leq i \leq n-3)$, and we will have positive quadratic hyponormality. We therefore turn to achieving this non-negativity in the three cases corresponding to the values of i in (2).

I. Work for $c(n,n) \geq 0$ $(n \geq 4)$. Using (1) and $c(n-1,n) = v_0 \cdots v_{n-1}$ we get

$$\begin{aligned} c(n,n) &= u_n c(n-1,n) + v_n \cdots v_4 \delta(2) \\ &= u_n v_0 \cdots v_{n-1} + v_4 \cdots v_n \delta(2) \\ &= \begin{cases} v_4 \cdots v_{n-1}(u_n v_0 v_1 v_2 v_3 + v_n \delta(2)) & (n > 4), \\ u_4 v_0 v_1 v_2 v_3 + v_4 \delta(2) & (n = 4). \end{cases} \end{aligned} \tag{3}$$

Dividing by u_n (respectively, u_4) and with $z_n = v_n/u_n$ (which is known to be increasing in n) we get

$$c(n,n) \geq 0 \ \text{ for } n \geq 4 \Longleftrightarrow v_0 v_1 v_2 v_3 + z_n \delta(2) \geq 0 \quad (n \geq 4).$$

There are two cases, depending on whether the sequence

$$E(z_n) := v_0 v_1 v_2 v_3 + z_n \delta(2)$$

is increasing (i.e., $\delta(2) \geq 0$) or decreasing (i.e., $\delta(2) < 0$).

Case I-A. $E(z_n)$ increasing in n (i.e., $\delta(2) \geq 0$). In this case $c(n,n) \geq 0$ $(n \geq 4)$ automatically.

Case I-B. $E(z_n)$ decreasing in n (i.e., $\delta(2) < 0$). In this case,

$$c(n,n) \geq 0 \ \ (n \geq 4) \Longleftrightarrow E(z_n) \geq 0 \ \ (n \geq 4) \Longleftrightarrow E(K) \geq 0,$$

where

$$\begin{aligned} K &= \lim z_n = -\frac{\Psi_1^2}{2\Psi_0}\left(\Psi_1 + \sqrt{\Psi_1^2 + 4\Psi_0}\right) \quad \text{(by [6, p.397])} \\ &= -\frac{\Psi_1^2}{\Psi_0} \quad \text{(because the norm is one [6, p. 392])} \\ &= \frac{(ab-1)^2}{a(a-1)(b-1)} \end{aligned}$$

with

$$\begin{aligned} \Psi_1 &= \frac{b(c-a)}{b-a} = \frac{-1+ab}{-1+a}, \\ \Psi_0 &= \frac{-ab(c-b)}{b-a} = \frac{a-ab}{-1+a}, \end{aligned}$$

due to the special choice of c.

II. Work for $c(n,n-1) \geq 0 \quad (n \geq 5)$. Using (1) we get for $n \geq 5$,

$$\begin{aligned} c(n,n-1) &= u_n c(n-1,n-1) + v_n \cdots v_4 \delta(1) \\ &= u_n[u_{n-1}v_0 \cdots v_{n-2} + v_4 \cdots v_{n-1}\delta(2)] + v_n \cdots v_4\delta(1) \\ &= (v_4 \cdots v_{n-2})[u_n u_{n-1} v_0 v_1 v_2 v_3 + u_n v_{n-1}\delta(2) + v_n v_{n-1}\delta(1)] \\ &= (v_4 \cdots v_{n-2})(u_n u_{n-1})[v_0 v_1 v_2 v_3 + z_{n-1}\delta(2) + z_{n-1}z_n\delta(1)]. \end{aligned}$$

So $c(n,n-1) \geq 0$ $(n \geq 5)$ is equivalent to

$$v_0 v_1 v_2 v_3 + z_{n-1}\delta(2) + z_{n-1}z_n\delta(1) \geq 0 \quad (n \geq 5).$$

Let

$$F(z_{n-1}, z_n) := v_0 v_1 v_2 v_3 + z_{n-1}\delta(2) + z_{n-1}z_n\delta(1).$$

Then

$$\begin{aligned} &F(z_n, z_{n+1}) - F(z_{n-1}, z_n) \\ &\quad = (z_n - z_{n-1})\delta(2) + (z_{n+1}z_n - z_n z_{n-1})\delta(1) \\ &\quad = (z_n - z_{n-1})[\delta(2) + \frac{z_{n+1}z_n - z_n z_{n-1}}{z_n - z_{n-1}}\delta(1)] \quad (n \geq 5). \end{aligned} \tag{4}$$

Now

$$\begin{aligned} \frac{z_{n+1}z_n - z_n z_{n-1}}{z_n - z_{n-1}} &= z_{n-1} + (z_n - z_{n-1}) + z_{n-1}\frac{z_{n+1} - z_n}{z_n - z_{n-1}} + (z_{n+1} - z_n) \\ &= -\frac{(1-ab)^3}{a(a-1)^2(b-1)} \quad (n \geq 5) \end{aligned}$$

is constant for $n \geq 5$ using Lemma 4.3 in [8] (slightly improved to $n \geq 5$). So either $F(z_{n-1}, z_n)$ is increasing for all $n \geq 5$ if $[\,\cdot\,]$ in (4) is positive or decreasing for all $n \geq 5$ if $[\,\cdot\,]$ in (4) is negative. Note that in the positive case, we may use

$$\begin{aligned} \frac{z_{n+1}z_n - z_n z_{n-1}}{z_n - z_{n-1}} &= \frac{z_{n+1}z_n - z_{n+1}z_{n-1}}{z_n - z_{n-1}} + \frac{z_{n+1}z_{n-1} - z_n z_{n-1}}{z_n - z_{n-1}} \\ &= z_{n+1} + z_{n-1}\frac{z_{n+1} - z_n}{z_n - z_{n-1}} \geq z_{n+1} \end{aligned}$$

and that

$$-\frac{(1-ab)^3}{a(a-1)^2(b-1)} > 0$$

to conclude that if $\delta(1) < 0$, then

$$\delta(2) + z_n\delta(1) \geq \delta(2) + z_{n+1}\delta(1) \geq \delta(2) + \frac{z_{n+1}z_n - z_n z_{n-1}}{z_n - z_{n-1}}\delta(1) \geq 0,$$

and it follows easily that $F(z_4, z_5) \geq 0$ and hence $c(5,4) \geq 0$ automatically.

In the case $F(z_{n-1}, z_n)$ is decreasing, by the usual argument $z_n \nearrow K$ what is required is $F(K,K) \geq 0$. So, summarizing, there are two cases, depending on $F(z_{n-1}, z_n)$ increasing or decreasing, where

$$F(z_{n-1}, z_n) = v_0 v_1 v_2 v_3 + z_{n-1}\delta(2) + z_{n-1}z_n\delta(1).$$

Case II-A. The sequence $F(z_{n-1}, z_n)$ is increasing. This is equivalent to

$$\delta(2) - \frac{(1-ab)^3}{a(a-1)^2(b-1)} \cdot \delta(1) \geq 0.$$

Then

$$c(n, n-1) \geq 0 \quad (n \geq 5) \Longleftrightarrow \begin{cases} \text{automatic if } \delta(1) \leq 0, \\ c(5,4) \geq 0 \text{ if } \delta(1) \geq 0. \end{cases}$$

Case II-B. The sequence $F(z_{n-1}, z_n)$ is decreasing. This is equivalent to

$$\delta(2) - \frac{(1-ab)^3}{a(a-1)^2(b-1)} \cdot \delta(1) < 0.$$

Then

$$\begin{aligned} & c(n, n-1) \geq 0 \quad (n \geq 5) \Longleftrightarrow F(K, K) \geq 0 \\ & \qquad \Longleftrightarrow v_0 v_1 v_2 v_3 + \frac{(1-ab)^2}{a(a-1)(b-1)} \cdot \delta(2) + \left[\frac{(1-ab)^2}{a(a-1)(b-1)}\right]^2 \cdot \delta(1) \geq 0. \end{aligned}$$

III. Work for $c(n, n-2) \geq 0 \quad (n \geq 5)$. We will check $c(5,3) \geq 0$ individually. For $n \geq 6$, using (1) and (3) we have that for $n \geq 6$,

$$\begin{aligned} c(n, n-2) &= u_n c(n-2, n-2) + v_n \cdots v_4 \delta(0) \\ &= u_n [u_{n-1} c(n-2, n-1) + v_{n-1} \cdots v_4 \cdot \delta(1)] + v_n \cdots v_4 \cdot \delta(0) \\ &= u_n u_{n-1} c(n-2, n-1) + u_n v_{n-1} \cdots v_4 \delta(1) + v_n \cdots v_4 \delta(0) \\ &= u_n u_{n-1} [u_{n-2} v_0 \cdots v_{n-3} + v_4 \cdots v_{n-2} \delta(2)] + u_n v_{n-1} \cdots v_4 \delta(1) \\ &\qquad + v_n \cdots v_4 \delta(0) \\ &= u_n u_{n-1} u_{n-2} v_0 \cdots v_{n-3} + u_n u_{n-1} v_{n-2} \cdots v_4 \delta(2) \\ &\quad + u_n v_{n-1} v_{n-2} \cdots v_4 \delta(1) + v_n \cdots v_4 \delta(0) \\ &= (u_n u_{n-1} u_{n-2})(v_4 \cdots v_{n-3}) \\ &\quad \times [v_0 v_1 v_2 v_3 + z_{n-2} \delta(2) + z_{n-2} z_{n-1} \delta(1) + z_{n-2} z_{n-1} z_n \delta(0)]. \end{aligned}$$

So the condition $c(n, n-2) \geq 0$ $(n \geq 5)$ is equivalent to the conditions $c(5,3) \geq 0$ and

$$\begin{aligned} G(z_{n-2}, z_{n-1}, z_n) := v_0 v_1 v_2 v_3 + z_{n-2} \delta(2) + z_{n-2} z_{n-1} \delta(1) \\ + z_{n-2} z_{n-1} z_n \delta(0) \geq 0 \quad (n \geq 6). \end{aligned}$$

Lemma 3.1. *For $n \geq 6$, we have that*

$$\frac{z_n z_{n-1}(z_{n+1} - z_{n-2})}{z_{n-1} - z_{n-2}} = \frac{(-1+ab)^4 (1 - a(1+b) + a^2(1 - b + b^2))}{a^2 (a-1)^4 (b-1)^2} > 0. \tag{5}$$

Proof. Let

$$J_n(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge = \frac{z_n z_{n-1}(z_{n+1} - z_{n-2})}{z_{n-1} - z_{n-2}},$$

where the z_n are those pertaining to $(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$. It is a computation to show

$$J_7(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge = J_6(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge = \Delta_{a,b},$$

where $\Delta_{a,b}$ is the middle term in (5). Recall next that, if $\sqrt{d}$ is the next weight in the sequence $(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$, $(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge = \sqrt{a}, (\sqrt{b}, \sqrt{c}, \sqrt{d})^\wedge$. Further, in notation whose meaning is obvious,

$$z_n(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge = z_{n-1}(\sqrt{b}, \sqrt{c}, \sqrt{d})^\wedge, \quad n \geq 5.$$

By what has been proved already,

$$\begin{aligned} J_8(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge &= J_7(\sqrt{b}, \sqrt{c}, \sqrt{d})^\wedge \\ &= J_6(\sqrt{b}, \sqrt{c}, \sqrt{d})^\wedge \\ &= J_7(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge \\ &= \Delta_{a,b}. \end{aligned}$$

Repeating this argument the result follows.

Now we have

$$\begin{aligned} &G(z_{n-1}, z_n, z_{n+1}) - G(z_{n-2}, z_{n-1}, z_n) \\ &= (z_{n-1} - z_{n-2}) \left[\delta(2) + \frac{z_{n-1}z_n - z_{n-2}z_{n-1}}{z_{n-1} - z_{n-2}} \delta(1) + \frac{z_n z_{n-1}(z_{n+1} - z_{n-2})}{z_{n-1} - z_{n-2}} \delta(0) \right] \\ &= (z_{n-1} - z_{n-2}) \\ &\quad \times \left[\delta(2) - \frac{(1-ab)^3}{a(a-1)^2(b-1)} \delta(1) + \frac{(ab-1)^4(1-a(1+b)+a^2(1-b+b^2))}{a^2(-1+a)^4(-1+b)^2} \cdot \delta(0) \right], \end{aligned}$$

for $n \geq 6$, and the second term $[\,\cdot\,]$ in the above equation is constant for $n \geq 6$. So there are two cases as usual, for G increasing or decreasing. Call the term in $[\,\cdot\,]$ above C_3 for convenience.

Case III-A. The sequence $G(z_{n-2}, z_{n-1}, z_n)$ is increasing $(n \geq 6)$. This is equivalent to $C_3 \geq 0$. Then

$$c(n, n-2) \geq 0 \quad (n \geq 5) \Longleftrightarrow c(5,3) \geq 0 \text{ and } c(6,4) \geq 0.$$

(In some cases, depending on signs of $\delta(0), \delta(1)$, and $\delta(2)$, $c(6,4) \geq 0$ may be automatic. We omit the details of these special cases.)

Case III-B. The sequence $G(z_{n-2}, z_{n-1}, z_n)$ is decreasing $(n \geq 6)$. This is equivalent to $C_3 \leq 0$. Then

$$\begin{aligned} c(n, n-2) &\geq 0 \quad (n \geq 5) \Longleftrightarrow \\ c(5,3) &\geq 0 \text{ and } G(K,K,K) = v_0v_1v_2v_3 + K\delta(2) + K^2\delta(1) + K^3\delta(0) \geq 0, \end{aligned}$$

where $K = \frac{(ab-1)^2}{(a-1)a(b-1)}$ as before.

We now arrive at the following theorem.

Theorem 3.2. *Let* $\widehat{\alpha}(x,y) : \sqrt{y}, \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c_{\text{norm}}})^{\wedge}$ *with* $0 < y \le x < a < b < 1$. *Then* $W_{\widehat{\alpha}(x,y)}$ *is positively quadratically hyponormal if and only if the following assertions hold*:

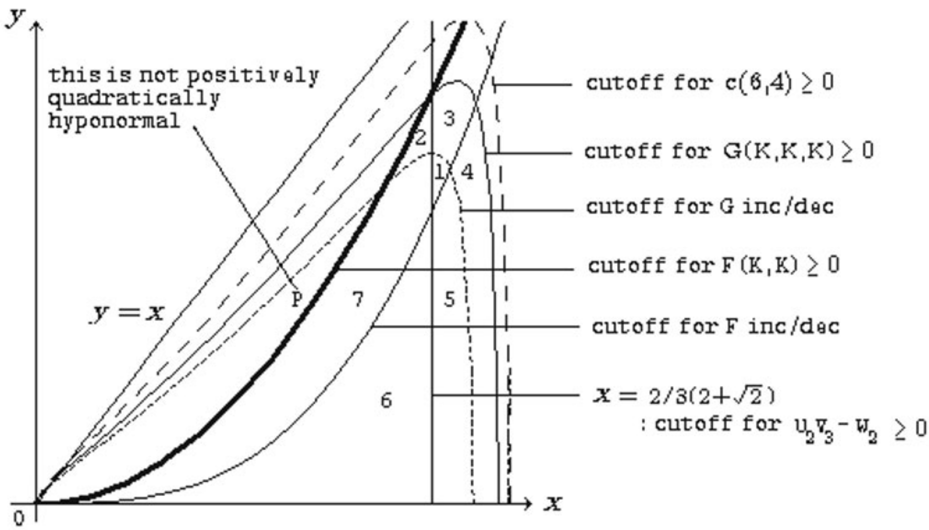
(i) $\delta(0) \ge 0, c(3,1), c(3,2), c(3,3), c(4,2), c(4,3), c(5,3) \ge 0$,
(ii) ($c(n,n)$ condition) I-A *or* I-B *holds,*
(iii) ($c(n,n-1)$ condition) II-A *or* II-B *holds* (II-A *requires checking* $c(5,4) \ge 0$), *and*
(iv) ($c(n,n-2)$ condition) III-A *or* II-B *holds* (III-A *requires checking* $c(6,4) \ge 0$).

Example 3.3. Consider $\widehat{\alpha}(x,y) : \sqrt{y}, \sqrt{x}, (\sqrt{\frac{1}{2+\sqrt{2}}}, \sqrt{\frac{2}{2+\sqrt{2}}}, \sqrt{\frac{3}{2+\sqrt{2}}})^{\wedge}$, which yields a shift of norm one by a simple computation. Then the region

$$\mathcal{R} := \{(x,y) : W_{\widehat{\alpha}(x,y)} \text{ is positively quadratically hyponormal}\}$$

is the union of the regions 1–7 in Figure 3.1.

Figure 3.1



(The diagram is not to scale, with small areas enlarged.)

Note that for these values of a and b, $c(6,4) \ge 0$ controls also

$$c(3,1), c(3,2), c(3,3), c(4,2), c(4,3), c(5,3) \ge 0.$$

Let

$$\widehat{\beta}(x) : \sqrt{x}, \left(\sqrt{\frac{1}{2+\sqrt{2}}}, \sqrt{\frac{2}{2+\sqrt{2}}}, \sqrt{\frac{3}{2+\sqrt{2}}}\right)^{\wedge}$$

and let $W_{\widehat{\beta}(x)}$ be the associated weighted shift.

Then we may detect the following in the regions:

1. $W_{\widehat{\beta}(x)}$ is not 2-hyponormal; I-A, II-B, III-A hold for $W_{\widehat{\alpha}(x,y)}$ in the region,
2. $W_{\widehat{\beta}(x)}$ is 2-hyponormal; I-A, II-B, III-B hold for $W_{\widehat{\alpha}(x,y)}$ in the region,
3. $W_{\widehat{\beta}(x)}$ is not 2-hyponormal; I-A, II-B, III-B hold for $W_{\widehat{\alpha}(x,y)}$ in the region,
4. $W_{\widehat{\beta}(x)}$ is not 2-hyponormal; I-A, II-A, III-B hold for $W_{\widehat{\alpha}(x,y)}$ in the region,
5. $W_{\widehat{\beta}(x)}$ is not 2-hyponormal; I-A, II-A, III-A hold for $W_{\widehat{\alpha}(x,y)}$ in the region,
6. $W_{\widehat{\beta}(x)}$ is 2-hyponormal; I-A, II-A, III-A hold for $W_{\widehat{\alpha}(x,y)}$ in the region,
7. $W_{\widehat{\beta}(x)}$ is 2-hyponormal; I-A, II-B, III-A hold for $W_{\widehat{\alpha}(x,y)}$ in the region.

This example will be continued in Example 4.7, in which the point P will figure as well.

Remark 3.4. (i) In Example 3.3 condition I-B does not hold for any region of positive quadratic hyponormality. However, with $a = 3/4$, $b = 4/5$, and c as usual there is a region of positive quadratic hyponormality in x and y on which I-B holds.

(ii) For $0 < a < b < 1$ and c as usual, consider x chosen so that $u_2v_3 - w_2 = 0$ (the largest x for which the weight sequence $\sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ with c as usual induces a 2-hyponormal shift). The intervals in y for which $F(K,K)$ and $G(K,K,K)$ are nonnegative (for $\sqrt{y}, \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$) coincide. The significance of this observation, if any, is not known.

4. Quadratic hyponormality

In this section we will obtain a characterization for quadratic hyponormality of the back step extension of a weighted shift with recursive tail.

4.1. General back step extensions

We will consider back step extensions of length k;

$$^k\widehat{\alpha} : \sqrt{y_k}, \sqrt{y_{k-1}}, \ldots, \sqrt{y_1}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$$

with $0 < y_k \le y_{k-1} < \cdots < y_1 < a < b < c$. We will use ${}^ku_i, {}^kv_i, {}^kw_i, {}^kq_i, \ldots$ to denote the usual quantities associated with $W_{{}^k\widehat{\alpha}}$. However, to save writing, we will use u_i for ${}^1u_i, v_i$ for ${}^1v_i, \ldots$, so $u_i, v_i, w_i, p_i, q_i, \ldots$ have the same meaning as in [10]. Observe for later use that

$$\begin{aligned} {}^ku_i &= u_{i-k} && (i \ge k+1) \\ {}^kv_i &= v_{i-k} && (i \ge k+2) \\ {}^kw_i &= w_{i-k} && (i \ge k+1). \end{aligned} \tag{6}$$

Next we must revisit Lemmas 4.3 and 4.4 in [10] and their proofs. Recall that this is in the context of a back step extension, and that our chosen notation matches that in the paper [10] except that the prefixed weight in their paper is $\sqrt{x}$ while ours is $\sqrt{y_1}$. Let us consider

$$\begin{aligned} F_n &= F_n(x_0, x_1, \ldots, x_n, s) : \\ &= \sum_{i=0}^{n} u_i x_i^2 - 2s \sum_{i=0}^{n-1} \sqrt{w_i} x_i x_{i+1} + s^2 \sum_{i=0}^{n} v_i x_i^2 \\ &= f_2(x_0, x_1, x_2, s) + \\ &+ \left[\sum_{i=2}^{n-1} u_i x_i^2 - 2s \sum_{i=2}^{n-1} \sqrt{w_i} x_i x_{i+1} + s^2 \sum_{i=3}^{n} v_i x_i^2 + u_n x_n^2 \right], \end{aligned} \tag{7}$$

where

$$f_2 := f_2(x_0, x_1, x_2, s) = q_0 x_0^2 - 2r_0 x_0 x_1 + q_1 x_1^2 - 2r_1 x_1 x_2 + v_2 s^2 x_2^2$$

(see [10]). For brevity, let us use $g_n(x_0, x_1, x_2, \ldots, x_n, s)$ for the second term in the last expression of F_n in (7). (Observe that x_0 and x_1 do not actually occur in g_n.) With this notation what is really proved in [10, Lemma 4.3] is the following lemma.

Lemma 4.1. *For $n \geq 3$, the following statements are equivalent*:

(i) *for any $x_0, \ldots, x_n, s$ in $\mathbb{R}_+$, $F_n = f_2 + g_n \geq 0$;*
(ii) *for any x_0, x_1, x_2, s in $\mathbb{R}_+$,*

$$f_2(x_0, x_1, x_2, s) + \frac{u_2 x_2^2}{1 + z_3 t + \cdots + z_3 z_4 \cdots z_n t^{n-2}} \geq 0.$$

And [10, Lemma 4.4] becomes the following lemma.

Lemma 4.2. *For $n \geq 3$, the following statements are equivalent*:

(i) *$F_n = f_2 + g_n \geq 0$ for all $x_0, \ldots, x_n \in \mathbb{R}_+$ and $s > \sqrt{1/K}$;*
(ii) *$f_2(x_0, x_1, x_2, s) \geq 0$ for all $x_0, x_1, x_2 \in \mathbb{R}_+$ and $s > \sqrt{1/K}$.*

In fact, in the proofs nothing specific is used about $f_2(x_0, x_1, x_2, s)$ and so what is really obtained is that

Lemma 4.1′. *For any $f(x_0, x_1, x_2, s)$ and $n \geq 3$, the following statements are equivalent*:

(i′) *for any $x_0, \ldots, x_n, s$ in $\mathbb{R}_+$, $f + g_n \geq 0$;*
(ii′) *for any x_0, x_1, x_2, s in $\mathbb{R}_+$,*

$$f(x_0, x_1, x_2, s) + \frac{u_2 x_2^2}{1 + z_3 t + \cdots + z_3 z_4 \cdots z_n t^{n-2}} \geq 0.$$

Lemma 4.2′. *For any $f(x_0, x_1, x_2, s)$ and $n \geq 3$, the following statements are equivalent*:

(i′) *$f + g_n \geq 0$ for all $x_0, \ldots, x_n \in \mathbb{R}_+$ and $s > \sqrt{1/K}$;*
(ii′) *$f(x_0, x_1, x_2, s) \geq 0$ for all $x_0, x_1, x_2 \in \mathbb{R}_+$ and $s > \sqrt{1/K}$.*

Now define, for $x_{-k+1},\dots,x_{-1},x_0,\dots,x_n,s\in\mathbb{R}_+$,

$$
\begin{aligned}
{}^kF_n&(x_{-k+1},\dots,x_{-1},x_0,\dots,x_n,s):=\\
&=\sum_{i=-k+1}^{n}{}^ku_ix_i^2-2s\sum_{i=-k+1}^{n-1}\sqrt{{}^kw_i}x_ix_{i+1}+s^2\sum_{i=-k+1}^{n}{}^kv_ix_i^2\\
&=\left[\sum_{i=-k+1}^{1}{}^ku_ix_i^2-2s\sum_{i=-k+1}^{1}\sqrt{{}^kw_i}x_ix_{i+1}+s^2\sum_{i=-k+1}^{2}{}^kv_ix_i^2\right] \qquad (8)\\
&\qquad+\left[\sum_{i=2}^{n}{}^ku_ix_i^2-2s\sum_{i=2}^{n-1}\sqrt{{}^kw_i}x_ix_{i+1}+s^2\sum_{i=3}^{n}{}^kv_ix_i^2\right].
\end{aligned}
$$

For brevity, we write $f_k(x_{-k+1},\dots,x_0,x_1,x_2,s)$ $[{}^kg_n(x_0,x_1,\dots,x_n,s)$, resp.] for the first [second, resp.] term of the final expression of ${}^kF_n$ in (8). Now observe that using (6) we have

$$
{}^kg_n(x_0,x_1,\dots,x_n,s)=g_n(x_0,x_1,\dots,x_n,s)
$$

and citing Lemmas 4.1′ and 4.2′ we get

Lemma 4.3. For $n\ge k+2$, *the following statements are equivalent*:

(i) *for all* $x_{-k+1},\dots,x_0,\dots,x_n,s\in\mathbb{R}_+$, $f_k+{}^kg_n\ge 0$;

(ii) *for all* $x_{-k+1},\dots,x_0,x_1,x_2,s\in\mathbb{R}_+$,

$$
\begin{aligned}
f_k&(x_{-k+1},\dots,x_0,x_1,x_2,s)+\frac{u_2x_2^2}{1+z_3t+\cdots+z_3\cdots z_nt^{n-2}}\\
&=f_k(x_{-k+1},\dots,x_0,x_1,x_2,s)\\
&\quad+\frac{{}^ku_{k+2}x_2^2}{1+{}^kz_{3+k}t+{}^kz_{3+k}{}^kz_{k+4}t^2+\cdots+{}^kz_{3+k}\cdots{}^kz_{n+k}t^{n-2}}\ge 0.
\end{aligned}
$$

Also, by the same idea but citing Lemma 4.2′ we get

Lemma 4.4. *For* $n\ge k+2$, *the following statements are equivalent*:

(i) *for all* $x_{-k+1},\dots,x_0,\dots,x_n\in\mathbb{R}_+$ *and* $s>\sqrt{1/K}$,

$$
{}^kF_n(x_{-k+1},\dots,x_{-1},x_0,\dots,x_n,s)\ge 0;
$$

(ii) *for all* $x_{-k+1},\dots,x_0,\dots,x_{k+1}\in\mathbb{R}_+$ *and* $s>\sqrt{1/K}$,

$$
f_k(x_{-k+1},\dots,x_0,x_1,x_2,s)\ge 0.
$$

Recall from [10, Th. 3.3], with notation adjusted to our setting, that quadratic hyponormality of $W_{{}^k\widehat{\alpha}}$ is equivalent to the existence of a positive integer n_0 such that the expression ${}^kF_n(x_{-k+1},\dots,x_{-1},x_0,\dots,x_n,s)$ is non-negative for all non-negative real numbers $x_{-k+1},\dots,x_{-1},x_0,\dots,x_n,s$ and all $n\ge n_0$. Combining this fact with the above lemmas, we obtain the following theorem.

Theorem 4.5. *Let* ${}^k\widehat{\alpha}:\sqrt{y_k},\sqrt{y_{k-1}},\dots,\sqrt{y_1},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge$ *be a weight sequence and let* $W_{{}^k\alpha}$ *be the associated shift. Then the following three statements are equivalent*:

(i) $W_{{}^k\alpha}$ *is quadratically hyponormal;*

(ii) *for all* $n \geq k+2$ *and* $x_{-k+1}, \ldots, x_0, \ldots, x_n, s \in \mathbb{R}_+$,

$$ {}^kF_n(x_{-k+1}, \ldots, x_{-1}, x_0, \ldots, x_n, s) \geq 0; $$

(iii-a) *for all* $x_{-k+1}, \ldots, x_0, x_1, x_2 \in \mathbb{R}_+$ *and* $s \leq \sqrt{1/K}$,

$$ f_k(x_{-k+1}, \ldots, x_0, x_1, x_2, s) + \frac{{}^ku_{k+2}x_2^2}{1 + {}^kz_{3+k}t + \cdots + {}^kz_{3+k} \cdots {}^k z_{n+k}t^{n-2}} \geq 0; $$

(iii-b) *for all* $x_{-k+1}, \ldots, x_0, x_1, x_2 \in \mathbb{R}_+$ *and* $s > \sqrt{1/K}$,

$$ f_k(x_{-k+1}, \ldots, x_0, x_1, x_2, s) \geq 0. $$

In particular, if it holds that for all $x_{-k+1}, \ldots, x_0, x_1, x_2, s \in \mathbb{R}_+$,

$$ f_k(x_{-k+1}, \ldots, x_0, x_1, x_2, s) \geq 0, $$

then all of (i), (ii), *and* (iii-a,b) *hold.*

4.2. Reduction to 2 back step extension

To solve Problem 1.1, we need to reduce to the 2 back step extension from the general back step extension in Subsection 4.1. We return to the notation in which the back step extension is $\sqrt{y}, \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ instead of $\sqrt{y_2}, \sqrt{y_1}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$. For $k = 3$, the function

$$ \begin{aligned} f_3(x_{-1}, x_0, x_1, x_2, s) = {} & \left(y + xys^2\right) x_{-1}^2 + \left((x-y) + axs^2\right) x_0^2 \\ & + \left((a-x) + s^2(ab - xy)\right) x_1^2 + (bc - ax)\, s^2 x_2^2 \\ & - 2s\sqrt{y}xx_{-1}x_0 - 2s\sqrt{x}\,(a-y)\,x_0x_1 - 2s\sqrt{a}\,(b-x)\,x_1x_2 \end{aligned} $$

is a quadratic form with respect to $(x_{-1}, x_0, x_1, x_2)^t$. The corresponding symmetric matrix is

$$ A(t) := \begin{bmatrix} y + xyt & -\sqrt{yt}x & 0 & 0 \\ -\sqrt{yt}x & (x-y) + axt & -(a-y)\sqrt{xt} & 0 \\ 0 & -(a-y)\sqrt{xt} & (a-x) + (ab - xy)\,t & -(b-x)\sqrt{at} \\ 0 & 0 & -(b-x)\sqrt{at} & (bc - ax)\,t \end{bmatrix}, $$

where $t := |s|^2$. Hence $f_3(x_{-1}, x_0, x_1, x_2, s) \geq 0$ if and only if $A(t) \geq 0$ for $t \geq 0$. We now apply the Nested Determinant Test to detect $A(t) \geq 0$ for $t \geq 0$. Recall that $y < x < a < b$. The 2×2 upper determinant of $A(t)$ is

$$ y(x-y) + yx(a-y)t + ax^2yt^2 \geq 0, \qquad t \geq 0. $$

The 3 by 3 upper determinant of $A(t)$ is

$$ \begin{aligned} (a-x)(x-y)y + a(b-x)(x-y)yt + axy(ab - x^2 - by + xy)t^2 \\ + ax^2y(ab - xy)t^3 \end{aligned} $$

with all coefficients positive except possibly the t^2 one, but

$$\begin{aligned} ab - x^2 - by + xy &= b(a-y) - x(x-y) \\ &\geq b(x-y) - x(x-y) \\ &= (b-x)(x-y) \geq 0. \end{aligned}$$

So the 3×3 determinant is positive for all $t \geq 0$. And finally

$$\det A(t) := ty(p_0 + p_1 t + p_2 t^2 + p_3 t^3),$$

where

$$p_0 = -(ab(b-c-2x) + a^2x + bcx)(x-y),$$

$$\begin{aligned} p_1 = &-a^2b^2x + ab^2cx + a^2bx^2 - abcx^2 - ab^2cy \\ &+ a^2bxy + ab^2xy + abcxy - a^2x^2y - 2abx^2y + ax^3y, \end{aligned}$$

$$\begin{aligned} p_2 = &\, a^2b^2cx - a^3bx^2 - a^2b^2x^2 + 2a^2bx^3 - abcx^3 - ab^2cxy \\ &+ a^2bx^2y + abcx^2y - a^2x^3y, \end{aligned}$$

$$p_3 = ax^2(bc - ax)(ab - xy).$$

So it is sufficient for quadratic hyponormality to have

$$\phi(x, y, t) := p_0 + p_1 t + p_2 t^2 + p_3 t^3 \geq 0 \qquad \text{for all } t \geq 0. \tag{9}$$

(Observe that this condition is not necessary since for $t < 1/K$ we need not have $f_3(x_{-1}, x_0, x_1, x_2, s) \geq 0$ all by itself.) Hence we have the following proposition.

Proposition 4.6. *If $\phi(x, y, t) \geq 0$ $(t \geq 0)$ as in (9), then $W_{\widehat{\alpha}(x,y)}$ is quadratically hyponormal.*

The following example gives a negative answer for Problem 1.1.

Example 4.7 (Continued from Example 3.3). Let $\widehat{\alpha}(x, y)$ be the weight sequence yielding the associated weighted shift $W_{\widehat{\alpha}(x,y)}$ as in Example 3.3. Take $x_0 = \frac{3}{5(2+\sqrt{2})}$ and $y_0 = \frac{27}{50(2+\sqrt{2})}$. Then $\phi(x_0, y_0, t) = \frac{6}{31250(2+\sqrt{2})^{10}}(a_0 + a_1 t + a_2 t^2 + a_3 t^3)$, where $a_0 = 49500 + 35000\sqrt{2}$, $a_1 = 105560 + 74620\sqrt{2}$, $a_2 = 764490 + 539640\sqrt{2}$, $a_3 = 339390 + 237573\sqrt{2}$. Hence obviously $\phi(x_0, y_0, t) > 0$ for all $t \geq 0$. By Proposition 4.6, $W_{\widehat{\alpha}(x_0,y_0)}$ is quadratically hyponormal. But by Example 3.3, $W_{\widehat{\alpha}(x_0,y_0)}$ is not positively quadratically hyponormal, because the point P in Figure 3.1 is $P = (x_0, y_0)$.

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Relaxed Commutant Lifting: An Equivalent Version and a New Application

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Abstract. This paper presents a few additions to commutant lifting theory. An operator interpolation problem is introduced and shown to be equivalent to the relaxed commutant lifting problem. Using this connection a description of all solutions of the former problem is given. Also a new application, involving bounded operators induced by H^2 operator-valued functions, is presented.

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0. Introduction

Let $\mathcal{U}$ and $\mathcal{Y}$ be Hilbert spaces, and let $\mathcal{F}$ be a subspace of $\mathcal{U}$. In this paper we consider the following problem. Given a contraction

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \tag{0.1}$$

find a (all) contraction(s) Γ from $\mathcal{U}$ into $H^2(\mathcal{Y})$ satisfying the equation

$$E_{\mathcal{Y}}\omega_1 + S_{\mathcal{Y}}\Gamma\omega_2 = \Gamma|_{\mathcal{F}}. \tag{0.2}$$

Here and in the sequel we use the convention that for any Hilbert space $\mathcal{N}$ the symbol $S_{\mathcal{N}}$ denotes the forward shift on the Hardy space $H^2(\mathcal{N})$ and $E_{\mathcal{N}}$ denotes the embedding of $\mathcal{N}$ into $H^2(\mathcal{N})$ defined by $(E_{\mathcal{N}}\, n)(\lambda) \equiv n$. Furthermore, $\Gamma|_{\mathcal{F}}$ stands for the restriction of Γ to $\mathcal{F}$ viewed as an operator from $\mathcal{F}$ into $H^2(\mathcal{Y})$.

A contraction Γ from $\mathcal{U}$ into $H^2(\mathcal{Y})$ satisfying equation (0.2) will be called a *solution to the interpolation problem defined by the contraction* ω in (0.1).

We shall show that the problem stated above can be reformulated as a relaxed commutant lifting problem. On the other hand, as we know from [4], the relaxed commutant lifting problem can be reduced to an interpolation problem defined by a special contraction ω of the form (0.1). Hence it follows that the above problem and the relaxed commutant lifting problem are equivalent in the sense that the one problem can be reduced to the other and conversely.

To state the main results we need some additional notation. Throughout $\mathcal{L}(\mathcal{U},\mathcal{Y})$ stands for the space of all (bounded linear) operators from $\mathcal{U}$ into $\mathcal{Y}$. By $\mathbf{H}^2(\mathcal{U},\mathcal{Y})$ we denote the space of all $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued functions that are analytic on $\mathbb{D}$ such that the Taylor coefficients $H_0, H_1, H_2, \ldots$ of the function H at zero satisfy the constraint $\sum_{n=0}^{\infty} \|H_n u\|^2 < \infty$ for each $u \in \mathcal{U}$. Given such a function H, the formula

$$(\Gamma u)(\lambda) = H(\lambda)u, \quad \lambda \in \mathbb{D}, \quad u \in \mathcal{U}, \tag{0.3}$$

defines an operator Γ from $\mathcal{U}$ into the Hardy space $H^2(\mathcal{Y})$, which we shall refer to as the *operator defined by* H. Conversely, if Γ is an operator from $\mathcal{U}$ into $H^2(\mathcal{Y})$, then there is a unique $H \in \mathbf{H}^2(\mathcal{U},\mathcal{Y})$ such that (0.3) holds, and in this case we call H the *defining function* of Γ.

Replacing Γ in (0.2) by its defining function we see that our problem has the following alternative formulation: find all $H \in \mathbf{H}^2(\mathcal{U},\mathcal{Y})$ satisfying

$$\omega_1 + \lambda H(\lambda)\omega_2 = H(\lambda)|_{\mathcal{F}}, \quad \lambda \in \mathbb{D}, \tag{0.4}$$

and such that the operator defined by H is a contraction. In this case we also say that H is a *solution to the interpolation problem defined by the contraction* ω in (0.1).

The connection with relaxed commutant lifting mentioned above allows us to use Theorem 1.1 in [6] (cf., Theorem 0.1 in [5]) to prove the following theorem.

Theorem 0.1. *Let ω be a contraction as in* (0.1)*. Then $H \in \mathbf{H}^2(\mathcal{U},\mathcal{Y})$ is a solution to the interpolation problem defined by the contraction ω if and only if H is given by*

$$H(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1}, \quad \lambda \in \mathbb{D}, \tag{0.5}$$

where Z is an arbitrary Schur class function from $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ satisfying the constraint $Z(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$. Here $\Pi_{\mathcal{Y}}$ and $\Pi_{\mathcal{U}}$ are the orthogonal projections from the Hilbert space direct sum $\mathcal{Y}\oplus\mathcal{U}$ onto $\mathcal{Y}$ and $\mathcal{U}$, respectively.

Recall that for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ the Schur class $\mathbf{S}(\mathcal{H},\mathcal{K})$ consists of all $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued functions F, analytic on $\mathbb{D}$, such that $\sup_{\lambda\in\mathbb{D}} \|F(\lambda)\| \leq 1$.

In general, the map $Z \mapsto H$ defined by Theorem 0.1 is not one-to-one. In fact, using the connection with relaxed commutant lifting and Theorem 1.2 in [6] we shall derive the following result.

Theorem 0.2. *Let $H \in \mathbf{H}^2(\mathcal{U},\mathcal{Y})$ be a solution to the interpolation problem defined by the contraction ω in* (0.1)*, and let Γ from $\mathcal{U}$ into $H^2(\mathcal{Y})$ be the operator defined by H. Then the set of all $Z \in \mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ satisfying $Z(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$ and such that* (0.5) *holds is parameterized by the set*

$$\big\{C \in \mathbf{S}(\mathcal{D}_\Gamma,\mathcal{D}_\Gamma) \mid C(\lambda)D_\Gamma|_{\mathcal{F}} = D_\Gamma\omega_2 \textit{ for each } \lambda \in \mathbb{D}\big\}. \tag{0.6}$$

Here D_Γ is the defect operator defined by the contraction Γ and $\mathcal{D}_\Gamma$ is the corresponding defect space.

The parameterization referred to in the preceding theorem can be made more explicit. Indeed, let $H \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ be a solution to the interpolation problem defined by the contraction ω in (0.1), and let Γ be the operator defined by H. Then Γ is a contraction. Given $C \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ we define

$$Z_C(\lambda) = \begin{bmatrix} 2H(\lambda) \\ \lambda^{-1}(W(\lambda) - I) \end{bmatrix} (W(\lambda) + I)^{-1}, \quad \lambda \in \mathbb{D}, \tag{0.7}$$

where

$$W(\lambda) = \Gamma^*(I + \lambda S_\mathcal{Y}^*)(I - \lambda S_\mathcal{Y}^*)^{-1}\Gamma + \\ + D_\Gamma(I + \lambda C(\lambda))(I - \lambda C(\lambda))^{-1} D_\Gamma, \quad \lambda \in \mathbb{D}. \tag{0.8}$$

We shall see that the map $C \mapsto Z_C$ induces a one-to-one map from the set (0.6) onto the set of all $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ satisfying $Z(\lambda)|_\mathcal{F} = \omega$ for each $\lambda \in \mathbb{D}$ and such that (0.5) holds.

As a new application we shall use Theorem 0.1 to prove the following result.

Theorem 0.3. *Let $H \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$, and let $\Theta \in \mathbf{S}(\mathcal{E}, \mathcal{U})$ be inner such that $\Theta(0) = 0$. Put $\mathcal{H} = H^2(\mathcal{U}) \ominus \Theta H^2(\mathcal{E})$. In order that the map $f \mapsto Hf$ defines a contraction from $\mathcal{H}$ into $H^2(\mathcal{Y})$ it is necessary and sufficient that H is given by*

$$H(\lambda) = \Pi_\mathcal{Y} Z(\lambda)(I - \Theta(\lambda)\Pi_\mathcal{E} Z(\lambda))^{-1}, \quad \lambda \in \mathbb{D}, \tag{0.9}$$

where Z is an arbitrary Schur class function from $\mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{E})$.

For $\Theta(\lambda) = \lambda^N$ the matrix-valued version of the above theorem can been found in [1, 2] and for operator-valued functions in [7]. For the scalar case, with $\Theta(\lambda) = \lambda$, the result goes back to [9], page 490.

The paper consist of three sections (not counting the present introduction). The first section has a preliminary character. Here we recall how the relaxed commutant lifting problem can be reduced to an interpolation problem of the type defined above. In the second section we prove Theorems 0.1 and 0.2 using relaxed commutant lifting. In the third section Theorem 0.3 is proved.

1. Preliminaries about relaxed commutant lifting

This section has a preliminary character. We recall the relaxed commutant lifting problem and how this problem can be reduced to an interpolation problem defined by a contraction of the form (0.1).

We begin with some terminology. A quintet $\{A, T', U', R, Q\}$ consisting of five Hilbert space operators is called a *data set* if the operator A is a contraction mapping $\mathcal{H}$ into $\mathcal{H}'$, the operator U' on $\mathcal{K}'$ is a minimal isometric lifting of the contraction T' on $\mathcal{H}'$, and R and Q are operators from $\mathcal{H}_0$ to $\mathcal{H}$ satisfying the following constraints:

$$T'AR = AQ \quad \text{and} \quad R^*R \le Q^*Q. \tag{1.1}$$

Without loss of generality we can and shall assume that U' is the Sz.-Nagy-Schäffer (minimal) isometric lifting of T'. The latter means (see [8]) that

$$U' = \begin{bmatrix} T' & 0 \\ E_{\mathcal{D}_{T'}} D_{T'} & S_{\mathcal{D}_{T'}} \end{bmatrix} \text{ on } \mathcal{K}' = \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}. \tag{1.2}$$

Given this data set the relaxed commutant lifting problem (*RCL problem*) is to find all contractions B from $\mathcal{H}$ to $\mathcal{K}'$ such that

$$\Pi_{\mathcal{H}'} B = A \quad \text{and} \quad U'BR = BQ. \tag{1.3}$$

Here $\Pi_{\mathcal{H}'}$ is the orthogonal projection from $\mathcal{K}'$ onto $\mathcal{H}'$ viewed as an operator from $\mathcal{K}'$ into $\mathcal{H}'$. In this case we refer to B as a *solution to the RCL problem* for the data set $\{A, T', U', R, Q\}$.

Since U' is given by (1.2), we have $\mathcal{K}' = \mathcal{H}' \oplus H^2(\mathcal{D}_{T'})$, and an operator B from $\mathcal{H}$ into $\mathcal{H}' \oplus H^2(\mathcal{D}_{T'})$ is a contraction satisfying the first identity in (1.3) if and only if B can be represented in the form

$$B = \begin{bmatrix} A \\ \Gamma D_A \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\ H^2(\mathcal{D}_{T'}) \end{bmatrix}, \tag{1.4}$$

where Γ is a contraction from $\mathcal{D}_A$ into $H^2(\mathcal{D}_{T'})$. Note that Γ and B in (1.4) define each other uniquely. Moreover, given (1.4) the second identity in (1.3) holds if and only if Γ satisfies the equation

$$E_{\mathcal{D}_{T'}} D_{T'} AR + S_{\mathcal{D}_{T'}} \Gamma D_A R = \Gamma D_A Q. \tag{1.5}$$

Therefore, with U' as in (1.2), the RCL problem for $\{A, T', U', R, Q\}$ is equivalent to the problem of finding all contractions Γ from $\mathcal{D}_A$ into $H^2(\mathcal{D}_{T'})$ such that (1.5) holds.

Equation (1.5) can be rewritten as an equation of the form (0.2). To see this one first observes that, because of (1.1), for each $h \in \mathcal{H}_0$ we have

$$\begin{aligned} \|D_A Qh\|^2 &= \|Qh\|^2 - \|AQh\|^2 \geq \|Rh\|^2 - \|T'ARh\|^2 \\ &= \|ARh\|^2 - \|T'ARh\|^2 + \|Rh\|^2 - \|ARh\|^2 \\ &= \|D_{T'}ARh\|^2 + \|D_A Rh\|^2. \end{aligned}$$

Hence the identity

$$\omega D_A Q h = \begin{bmatrix} D_{T'} ARh \\ D_A Rh \end{bmatrix}, \quad h \in \mathcal{H}_0,$$

uniquely defines a contraction ω from $\mathcal{F} = \overline{D_A Q \mathcal{H}}$ into $\mathcal{D}_{T'} \oplus \mathcal{D}_A$. We refer to this contraction as the *contraction underlying* the data set $\{A, T', U', R, Q\}$. Using this contraction equation (1.5) can equivalently be represented as

$$E_{\mathcal{D}_{T'}} \omega_1 + S_{\mathcal{D}_{T'}} \Gamma \omega_2 = \Gamma|_{\mathcal{F}}, \tag{1.6}$$

where ω_1 is the contraction mapping $\mathcal{F}$ into $\mathcal{D}_{T'}$ determined by the first component of ω and ω_2 is the contraction mapping $\mathcal{F}$ into $\mathcal{D}_A$ determined by the second component of ω. Summarizing the above discussion we arrive at the following conclusion.

With U' equal to the Sz.-Nagy-Schäffer isometric lifting of T', the RCL problem for $\{A, T', U', R, Q\}$ is equivalent to the problem of finding all contractions Γ from $\mathcal{D}_A$ into $H^2(\mathcal{D}_{T'})$ satisfying (1.6)*, where*

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{D}_{T'} \\ \mathcal{D}_A \end{bmatrix}$$

is the contraction underlying the given data set. Moreover, the map $\Gamma \mapsto B$ determined by (1.4) *provides a one-to-one correspondence between the solutions of the interpolation problem defined by ω and the solutions of the RCL problem for $\{A, T', U', R, Q\}$. In particular, any RCL problem reduces to a problem of the type considered in the introduction.*

2. Proofs of Theorems 0.1 and 0.2

Throughout this section $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, $\mathcal{F}$ is a subspace of $\mathcal{U}$, and the operator ω in (0.1) is a contraction. We associate with ω a lifting data set.

Proposition 2.1. *Let ω be a contraction as in* (0.1)*. Put*

$$\tilde{A} = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & 0 \end{bmatrix} \ \textit{on} \ \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \quad \tilde{T}' = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \ \textit{on} \ \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}$$

$$\tilde{R} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} 0 \\ \Pi_{\mathcal{F}}^* \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix},$$

$$\tilde{U}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{\mathcal{U}} & 0 \\ E_{\mathcal{Y}} & 0 & S_{\mathcal{Y}} \end{bmatrix} \ \textit{on} \ \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \\ H^2(\mathcal{Y}) \end{bmatrix}.$$

Then $\{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\}$ is a data set, and the underlying contraction is precisely the given contraction ω. Furthermore, $\tilde{U}'$ is the Sz.-Nagy-Schäffer isometric lifting of $\tilde{T}'$.

Here $\Pi_{\mathcal{F}}$ stands for the orthogonal projection of $\mathcal{U}$ onto $\mathcal{F}$ viewed as a map from $\mathcal{U}$ into $\mathcal{F}$, and hence $\Pi_{\mathcal{F}}^*$ is the canonical embedding of $\mathcal{F}$ into $\mathcal{U}$.

Proof. The operators $\tilde{A}$ and $\tilde{T}'$ are orthogonal projections and hence contractions. Observe that $\tilde{T}'\tilde{A}$ and $\tilde{A}\tilde{Q}$ are both zero operators. Furthermore, note that $\tilde{R} = \omega$ is a contraction defined on $\mathcal{F}$ and $\tilde{Q}^*\tilde{Q}$ is the identity operator on $\mathcal{F}$. From these remarks we see that

$$\tilde{T}'\tilde{A}\tilde{R} = \tilde{A}\tilde{Q} \quad \text{and} \quad \tilde{R}^*\tilde{R} \le \tilde{Q}^*\tilde{Q}. \tag{2.1}$$

Next, observe that

$$D_{\tilde{T}'} = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & 0 \end{bmatrix} \ \text{on} \ \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}.$$

Thus we can identify $\mathcal{D}_{\tilde{T}'}$ with the space $\mathcal{Y}$. With this identification in mind it is straightforward to check that $\tilde{U}'$ is the Sz.-Nagy-Schäffer isometric lifting of $\tilde{T}'$. It follows that $\{\tilde{A},\tilde{T}',\tilde{U}',\tilde{R},\tilde{Q}\}$ is a data set. Notice that in this case the space $\mathcal{H}_0$ appearing in the definition of a data set is equal to the space $\mathcal{F}$. Using

$$D_{\tilde{A}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix},$$

we see that

$$D_{\tilde{A}}\tilde{Q} = \tilde{Q}, \quad D_{\tilde{T}'}\tilde{A}\tilde{R} = \omega_1, \quad D_{\tilde{A}}\tilde{R} = \omega_2.$$

It is then easy to show that the contraction ω in (0.1) is precisely the contraction underlying the data set $\{\tilde{A},\tilde{T}',\tilde{U}',\tilde{R},\tilde{Q}\}$. □

Proof of Theorem 0.1. Let ω be a contraction as in (0.1), and let $\{\tilde{A},\tilde{T}',\tilde{U}',\tilde{R},\tilde{Q}\}$ be the data set constructed in Proposition 2.1. Since ω is the contraction underlying this data set and $\tilde{U}'$ is Sz.-Nagy-Schäffer isometric lifting of $\tilde{T}'$, we know (see the conclusion at the end of the previous section) that an operator $\Gamma : \mathcal{U} \to H^2(\mathcal{Y})$ is a solution of interpolation problem defined by the contraction ω if and only if the operator

$$B = \begin{bmatrix} \tilde{A} \\ \Gamma D_{\tilde{A}} \end{bmatrix} : \mathcal{Y}\oplus\mathcal{U} \to \begin{bmatrix} \mathcal{Y}\oplus\mathcal{U} \\ H^2(\mathcal{Y}) \end{bmatrix} \tag{2.2}$$

is a solution to the RCL problem for the data set $\{\tilde{A},\tilde{T}',\tilde{U}',\tilde{R},\tilde{Q}\}$. Recall (using canonical identifications) that $\mathcal{D}_{\tilde{T}'}$ and $\mathcal{D}_{\tilde{A}}$ are equal to $\mathcal{Y}$ and $\mathcal{U}$, respectively. But then Theorem 1.1 in [6] tells use that B in (2.2) is a solution to the RCL problem for the data set $\{\tilde{A},\tilde{T}',\tilde{U}',\tilde{R},\tilde{Q}\}$ if and only if the defining function H of Γ is given by

$$H(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda\Pi_{\mathcal{U}} Z(\lambda))^{-1}, \quad \lambda \in \mathbb{D},$$

where Z is an arbitrary Schur class function from $\mathbf{S}(\mathcal{U},\mathcal{Y}\oplus\mathcal{U})$ satisfying the constraint $Z(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$. From these two "if and only if" statements Theorem 0.1 follows. □

Proof of Theorem 0.2. Let $H \in \mathbf{H}^2(\mathcal{U},\mathcal{Y})$ be a solution to the interpolation problem defined by the contraction ω in (0.1), and let Γ from $\mathcal{U}$ into $H^2(\mathcal{Y})$ be the operator defined by H. Then Γ is a contraction satisfying (0.2). Hence for each $f \in \mathcal{F}$ we have

$$\begin{aligned} \|D_\Gamma f\|^2 &= \|f\|^2 - \|\Gamma f\|^2 = \|f\|^2 - \|E_{\mathcal{Y}}\omega_1 f\|^2 - \|S_{\mathcal{Y}}\Gamma\omega_2 f\|^2 \\ &= \|f\|^2 - \|\omega_1 f\|^2 - \|\omega_2 f\|^2 + \|\omega_2 f\|^2 - \|\Gamma\omega_2 f\|^2 \\ &\geq \|D_\Gamma \omega_2 f\|^2. \end{aligned} \tag{2.3}$$

The above calculation shows that there exists a (unique) contraction Ω mapping $\mathcal{F}_\Gamma = \overline{D_\Gamma\mathcal{F}}$ into $\mathcal{D}_\Gamma$ such that

$$\Omega D_\Gamma|_{\mathcal{F}} = D_\Gamma\omega_2. \tag{2.4}$$

Now let $\mathbf{S}'(\mathcal{D}_\Gamma,\mathcal{D}_\Gamma)$ be the set defined by (0.6). Let C be a function in $\mathbf{S}(\mathcal{D}_\Gamma,\mathcal{D}_\Gamma)$. Using the identity (2.4) we see that the function C belongs to $\mathbf{S}'(\mathcal{D}_\Gamma,\mathcal{D}_\Gamma)$ if and only if $C(\lambda)D_\Gamma|_\mathcal{F} = \Omega D_\Gamma|_\mathcal{F}$ for each $\lambda \in \mathbb{D}$. The latter identity can be rewritten as $C(\lambda)|_{\mathcal{F}_\Gamma} = \Omega$ for each $\lambda \in \mathbb{D}$. We conclude that

$$\mathbf{S}'(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma) = \big\{C \in \mathbf{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma) \mid C(\lambda)|_{\mathcal{F}_\Gamma} = \Omega \text{ for each } \lambda \in \mathbb{D}\big\}. \qquad (2.5)$$

In other words, for the data set considered here the set (0.6) is precisely the set $\mathbf{S}_\Omega(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ appearing in Theorem 1.2 of [6]. Next, recall that

$$B = \begin{bmatrix} \tilde{A} \\ \Gamma D_{\tilde{A}} \end{bmatrix} : \mathcal{Y} \oplus \mathcal{U} \to \begin{bmatrix} \mathcal{Y} \oplus \mathcal{U} \\ H^2(\mathcal{Y}) \end{bmatrix}$$

is a solution to the RCL problem for the data set $\{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\}$. But then, using that the set (0.6) is equal to the right-hand side of (2.5), we can apply Theorem 1.2 in [6] to complete the proof. □

Corollary 2.2. *Assume that ω in* (0.1) *is an isometry such that $\omega_2\mathcal{F}$ is dense in $\mathcal{U}$. Then the map $Z \mapsto H$ defined by Theorem* 0.1 *is one-to-one, and* (0.5) *provides a proper parameterization of all solutions to the interpolation problem defined by ω.*

Proof. Using the fact that ω is an isometry, we see from (2.3) that the operator Ω in (2.5) is also an isometry. In particular, the space $\Omega\mathcal{F}_\Gamma$ is a closed subspace of $\mathcal{D}_\Gamma$. Since $\omega_2\mathcal{F}$ is dense in $\mathcal{U}$, the space $D_\Gamma\omega_2\mathcal{F}$ is dense in $\mathcal{D}_\Gamma$. By (2.4) the space $D_\Gamma\omega_2\mathcal{F}$ is contained in $\Omega\mathcal{F}_\Gamma$. Thus $\Omega\mathcal{F}_\Gamma$ is also dense in $\mathcal{D}_\Gamma$. But $\Omega\mathcal{F}_\Gamma$ is closed in $\mathcal{D}_\Gamma$, and therefore $\Omega_\Gamma\mathcal{F} = \mathcal{D}_\Gamma$. Hence Ω is a unitary operator from $\mathcal{F}_\Gamma$ onto $\mathcal{D}_\Gamma$. This implies that the set defined by the right-hand side of (2.5), or equivalently the set (0.6), consists of one element only. Thus if ω is an isometry and $\omega_2\mathcal{F}$ is dense in $\mathcal{U}$, then the map $Z \mapsto H$ defined by Theorem 0.1 is one-to-one. □

From Theorem 1.2 in [6] it follows that the map $C \mapsto Z_C$ given by (0.7) and (0.8) is well defined and induces a one-to-one map from the set (0.6) onto the set of all $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U})$ satisfying $Z(\lambda)|_\mathcal{F} = \omega$ for each $\lambda \in \mathbb{D}$ and such that (0.5) holds.

3. Proof of Theorem 0.3

Throughout this section $H \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$, and $\Theta \in \mathbf{S}(\mathcal{E}, \mathcal{U})$ is inner such that $\Theta(0) = 0$. Furthermore, $\mathcal{H} = H^2(\mathcal{U}) \ominus \Theta H^2(\mathcal{E})$. Our aim is to prove Theorem 0.3. We begin with some auxiliary results.

Lemma 3.1. *Let $\Phi = \lambda^{-1}\Theta$, and put $\mathcal{H}_0 = H^2(\mathcal{U}) \ominus \Phi H^2(\mathcal{E})$. Then*

$$\mathcal{H} = E_\mathcal{U}\,\mathcal{U} \oplus \lambda\mathcal{H}_0, \quad \mathcal{H} = \mathcal{H}_0 \oplus \Phi E_\mathcal{E}\,\mathcal{E}. \qquad (3.1)$$

Proof. As usual, given any inner function $\alpha \in \mathbf{S}(\mathcal{X}, \mathcal{Y})$, we shall denote the space $H^2(\mathcal{Y}) \ominus \alpha H^2(\mathcal{X})$ by $H(\alpha)$. The two identities in (3.1), then follow from the rule (see, e.g., Theorem X.1.9 in [3]) that for two inner functions $\alpha \in \mathbf{S}(\mathcal{X}, \mathcal{Y})$ and $\beta \in \mathbf{S}(\mathcal{Y}, \mathcal{X})$ we have

$$H(\beta\alpha) = H(\beta) \oplus \beta H(\alpha).$$

Indeed, we apply this rule twice. First with $\alpha(\lambda) = \Psi(\lambda)$ and $\beta(\lambda) = \lambda I_{\mathcal{U}}$, and next with $\alpha(\lambda) = \lambda I_{\mathcal{E}}$ and $\beta(\lambda) = \Psi(\lambda)$. Note that in both cases $\beta\alpha = \Theta$. With the first choice of α and β we get the first identity in (3.1), and the second choice yields the second identity in (3.1). □

The above lemma allows us to define the following auxiliary operators:

$$R : \mathcal{H}_0 \to \mathcal{H}, \quad Rh_0 = h_0; \quad Q : \mathcal{H}_0 \to \mathcal{H}, \quad Qh_0 = \lambda h_0 \quad (h_0 \in \mathcal{H}_0). \tag{3.2}$$

Note that the operators R and Q are isometries.

Lemma 3.2. *Let $\Gamma : \mathcal{H} \to H^2(\mathcal{Y})$ be a (bounded linear) operator, and put*

$$K(\lambda) = E_{\mathcal{Y}}^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}\Gamma E_{\mathcal{U}}, \quad \lambda \in \mathbb{D}. \tag{3.3}$$

Then $K \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$, and the operator Γ satisfies $S_{\mathcal{Y}}\Gamma R = \Gamma Q$ if and only if for each $f \in \mathcal{H}$ we have

$$(\Gamma f)(\lambda) = K(\lambda)f(\lambda), \quad \lambda \in \mathbb{D}. \tag{3.4}$$

Proof. According to the first identity in (3.1) the operator $E_{\mathcal{U}}$ maps $\mathcal{U}$ into $\mathcal{H}$. Thus $\Gamma E_{\mathcal{U}}$ is well defined, and hence the same holds true for K. Obviously, K is an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function which is analytic on $\mathbb{D}$. Let K_n be the nth coefficient of the Taylor expansion of K at zero. Take $u \in \mathcal{U}$. Then $K_n u = E_{\mathcal{Y}}^*(S_{\mathcal{Y}}^*)^n \Gamma E_{\mathcal{U}}\, u = (\Gamma E_{\mathcal{U}}\, u)_n$, where $(\Gamma E_{\mathcal{U}}\, u)_n$ is the n-th coefficient of the Taylor expansion of the $\mathcal{Y}$-valued function $\Gamma E_{\mathcal{U}}\, u$ at zero. Therefore

$$(\Gamma E_{\mathcal{U}} u)(\lambda) = K(\lambda)u, \quad u \in \mathcal{U}, \ \lambda \in \mathbb{D}.$$

Thus $K \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$, and K is the defining function for the operator $\Gamma E_{\mathcal{U}}$.

Next assume that Γ satisfies the intertwining relation $S_{\mathcal{Y}}\Gamma R = \Gamma Q$. From (3.3) we see that (3.4) holds for $f = E_{\mathcal{U}}\, u$ with $u \in \mathcal{U}$ arbitrary. Indeed, for $f = E_{\mathcal{U}}\, u$ we have $f(\lambda) \equiv u$, and hence

$$K(\lambda)f(\lambda) = K(\lambda)u = E_{\mathcal{Y}}^*(I - \lambda S_{\mathcal{Y}}^*)^{-1}(\Gamma E_{\mathcal{U}}\, u) = (\Gamma E_{\mathcal{U}}\, u)(\lambda) = (\Gamma f)(\lambda).$$

Using the first equality in (3.1), we see that it suffices to prove (3.4) for $f = \lambda h_0$ with $h_0 \in \mathcal{H}_0$. However for such a function f we have

$$\Gamma f = \Gamma Q h_0 = S_{\mathcal{Y}}\Gamma R h_0 = S_{\mathcal{Y}}\Gamma h_0 = \lambda \Gamma h_0, \quad K(\lambda)f(\lambda) = \lambda K(\lambda)h_0(\lambda).$$

Therefore, it suffices to prove (3.4) for $h_0 \in \mathcal{H}_0$.

Take $h_0 \in \mathcal{H}_0$. Note that the identity operator on $H^2(\mathcal{U})$ is equal to $S_{\mathcal{U}}S_{\mathcal{U}}^* + P_{\mathcal{U}}$, where $P_{\mathcal{U}}$ is the orthogonal projection of $H^2(\mathcal{U})$ onto $E_{\mathcal{U}}\mathcal{U}$. The first identity in (3.1) shows that $E_{\mathcal{U}}\mathcal{U} \subset \mathcal{H}$. Thus $\Gamma P_{\mathcal{U}}$ is well defined. Since $\mathcal{H}_0$ is invariant under $S_{\mathcal{U}}^*$, we see that $S_{\mathcal{U}}S_{\mathcal{U}}^* h_0$ belongs to $\mathcal{H}$, and hence

$$\begin{aligned} \Gamma h_0 &= \Gamma S_{\mathcal{U}}S_{\mathcal{U}}^* h_0 + \Gamma P_{\mathcal{U}} h_0 = \Gamma Q S_{\mathcal{U}}^* h_0 + \Gamma P_{\mathcal{U}} h_0 \\ &= S_{\mathcal{Y}}\Gamma R S_{\mathcal{U}}^* h_0 + \Gamma P_{\mathcal{U}} h_0 = S_{\mathcal{Y}}\Gamma S_{\mathcal{U}}^* h_0 + \Gamma P_{\mathcal{U}} h_0. \end{aligned}$$

Since h_0 is an arbitrary element of $\mathcal{H}_0$, we get

$$\Gamma h_0 - S_{\mathcal{Y}}\Gamma S_{\mathcal{U}}^* h_0 = \Gamma P_{\mathcal{U}} h_0, \quad h_0 \in \mathcal{H}_0.$$

By induction, using $\mathcal{H}_0$ is invariant under $S^*_{\mathcal{U}}$, the preceding identity yields

$$\Gamma h_0 = \sum_{\nu=0}^{n} S^{\nu}_{\mathcal{Y}}(\Gamma P_{\mathcal{U}})(S^*_{\mathcal{U}})^{\nu} h_0 + S^{(n+1)}_{\mathcal{Y}}\Gamma(S^*_{\mathcal{U}})^{(n+1)} h_0, \quad h_0 \in \mathcal{H}_0,$$

for $n = 0, 1, 2, \ldots$. Now fix $\lambda \in \mathbb{D}$. From (3.3) we know that

$$(\Gamma P_{\mathcal{U}} f)(\lambda) = (\Gamma E_{\mathcal{U}} E^*_{\mathcal{U}} f)(\lambda) = K(\lambda) E^*_{\mathcal{U}} f, \quad f \in H^2(\mathcal{U}).$$

Hence for $h_0 \in \mathcal{H}_0$ we have

$$\begin{aligned}(\Gamma h_0)(\lambda) &= \sum_{\nu=0}^{n} \lambda^{\nu} K(\lambda) E^*_{\mathcal{U}} (S^*_{\mathcal{U}})^{\nu} h_0 + \lambda^{n+1}\big(\Gamma(S^*_{\mathcal{U}})^{(n+1)} h_0\big)(\lambda)\\ &= K(\lambda)\big(\sum_{\nu=0}^{n} \lambda^{\nu} E^*_{\mathcal{U}} (S^*_{\mathcal{U}})^{\nu} h_0\big) + \lambda^{n+1}\big(\Gamma(S^*_{\mathcal{U}})^{(n+1)} h_0\big)(\lambda). \qquad (3.5)\end{aligned}$$

Note that for $n \to \infty$ the function $\Gamma(S^*_{\mathcal{U}})^{(n+1)} h_0$ converges to zero in the norm of $H^2(\mathcal{Y})$, and hence the same holds true for $S^{(n+1)}_{\mathcal{Y}}\Gamma(S^*_{\mathcal{U}})^{(n+1)} h_0$. It follows that the second term in the right side of (3.5) converges to zero when $n \to \infty$. Furthermore, for $n \to \infty$ the vector $\sum_{\nu=0}^{n} \lambda^{\nu} E^*_{\mathcal{U}} (S^*_{\mathcal{U}})^{\nu} h_0$ converges to $h_0(\lambda)$ in the norm of $\mathcal{U}$. Hence the first term in the right-hand side of (3.5) converges to $K(\lambda)h_0(\lambda)$ when $n \to \infty$. Thus we have proved that (3.4) holds.

To prove the converse implication. Assume that (3.4) holds. Let $h_0 \in \mathcal{H}_0$. Then for each $\lambda \in \mathbb{D}$ we have

$$\begin{aligned}(\Gamma Q h_0)(\lambda) = K(\lambda)\lambda h_0(\lambda) = \lambda K(\lambda) h_0(\lambda) &= (S_{\mathcal{Y}}\Gamma h_0)(\lambda)\\ &= (S_{\mathcal{Y}}\Gamma R h_0)(\lambda).\end{aligned}$$

Since h_0 is an arbitrary element in $\mathcal{H}_0$, we see that $S_{\mathcal{Y}}\Gamma R = \Gamma Q$. □

Next we put the problem into the setting of our alternative version of the relaxed commutant lifting problem. Put $\mathcal{F} = \lambda\mathcal{H}_0$, and define

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{H} \end{bmatrix}, \quad \omega_1 = 0, \quad \omega_2(\lambda h_0) = h_0 \ (h_0 \in \mathcal{H}_0). \qquad (3.6)$$

Note that $\omega_2 Q = R$, and hence a contraction Γ from $\mathcal{H}$ into $H^2(\mathcal{Y})$ satisfies the intertwining relation $S_{\mathcal{Y}}\Gamma R = \Gamma Q$ if and only if $S_{\mathcal{Y}}\Gamma\omega_2 = \Gamma|_{\mathcal{F}}$. Since $\omega_1 = 0$, we are now ready to prove the main result.

Proof of Theorem 0.3. Let $H \in \mathbf{H}^2(\mathcal{U},\mathcal{Y})$, and let us assume that the map $f \mapsto Hf$ defines a contraction from $\mathcal{H}$ into $H^2(\mathcal{Y})$. Denote this contraction by Γ. Then (3.3) holds with H in place of K, and Lemma 3.2 shows that the contraction Γ satisfies the intertwining relation $S_{\mathcal{Y}}\Gamma R = \Gamma Q$, and thus $S_{\mathcal{Y}}\Gamma\omega_2 = \Gamma|_{\mathcal{F}}$. Since $\omega_1 = 0$, we know from Theorem 0.1 that H is given by

$$H(\lambda) = F(\lambda)(I_{\mathcal{H}} - \lambda G(\lambda))^{-1} E_{\mathcal{U}}, \quad \lambda \in \mathbb{D}, \qquad (3.7)$$

where

$$W = \begin{bmatrix} F \\ G \end{bmatrix} \in \mathbf{S}(\mathcal{H}, \begin{bmatrix} \mathcal{Y} \\ \mathcal{H} \end{bmatrix}), \quad W(\lambda)|_{\mathcal{F}} = \omega \quad (\lambda \in \mathbb{D}). \tag{3.8}$$

Conversely, let $H \in \mathbf{H}^2(\mathcal{U}, \mathcal{Y})$ be given by (3.7), where F and G are such that (3.8) holds. Then, again using Theorem 0.1, we know that there exists a contraction Γ from $\mathcal{H}$ into $H^2(\mathcal{Y})$ such that

$$(\Gamma h)(\lambda) = F(\lambda)(I_{\mathcal{H}} - \lambda G(\lambda))^{-1} h, \quad h \in \mathcal{H}, \ \lambda \in \mathbb{D}.$$

Moreover, $S_{\mathcal{Y}} \Gamma \omega_2 = \Gamma|_{\mathcal{F}}$, and hence Γ satisfies the intertwining relation $S_{\mathcal{Y}} \Gamma R = \Gamma Q$. It follows that $H(\lambda) = (\Gamma E_{\mathcal{U}})(\lambda)$ for each $\lambda \in \mathbb{D}$. The fact that Γ satisfies the intertwining relation $S_{\mathcal{Y}} \Gamma R = \Gamma Q$ allows us to again apply Lemma 3.2. We conclude that $(\Gamma h)(\lambda) = H(\lambda) h(\lambda)$ for each $h \in \mathcal{H}$ and each $\lambda \in \mathbb{D}$. Thus the map $f \mapsto Hf$ induces a contraction from $\mathcal{H}$ into $H^2(\mathcal{Y})$ as desired.

From the previous results we see that it remains to show that the representations given by (3.7), (3.8) and by (0.9), with $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{E})$, are equivalent. Consider the spaces

$$\mathcal{F} = \lambda \mathcal{H}_0, \ \mathcal{G} = \mathcal{H} \ominus \mathcal{F} = E_{\mathcal{U}} \mathcal{U}, \ \mathcal{F}' = \mathcal{H}_0, \ \mathcal{G}' = \mathcal{H} \ominus \mathcal{F}' = \Phi E_{\mathcal{E}} \mathcal{E}. \tag{3.9}$$

Let F and G be as in (3.8). Fix $z \in \mathbb{D}$. Since $G(z)|_{\mathcal{F}} = \omega_2$ and ω_2 is an isometry which maps $\mathcal{F}$ onto $\mathcal{F}'$, we know that $G(z)\mathcal{G} \subset \mathcal{G}' = \Phi E_{\mathcal{E}} \mathcal{E}$. Thus given $u \in \mathcal{U}$, we have $G(z) E_{\mathcal{U}} u = \Phi E_{\mathcal{E}} e(z)$ for some $e(z) \in \mathcal{E}$. Let M_Φ be the operator of multiplication by Φ acting from $H^2(\mathcal{E})$ onto $H^2(\mathcal{U})$. The fact that Φ is inner is equivalent to the statement that M_Φ in an isometry. Put $C(z) = E_{\mathcal{E}}^* M_\Phi^* G(z) E_{\mathcal{U}}$. Then

$$C(z)u = E_{\mathcal{E}}^* M_\Phi^* G(z) E_{\mathcal{U}} u = E_{\mathcal{E}}^* M_\Phi^* \Phi E_{\mathcal{E}} e(z) = E_{\mathcal{E}}^* E_{\mathcal{E}} e(z) = e(z).$$

We conclude that $G(z) E_{\mathcal{U}} = \Phi E_{\mathcal{E}} C(z)$. From the definition of $C(z)$ it is clear that $C(z)$ is a bounded linear operator from $\mathcal{U}$ into $\mathcal{E}$. Moreover, $C(z)$ depends analytically on $z \in \mathbb{D}$.

From the result of the previous paragraph we know that

$$G(z)(E_{\mathcal{U}} u \oplus \lambda h_0) = h_0 \oplus \Phi E_{\mathcal{E}} C(z) = E_{\mathcal{U}} v \oplus \lambda k_0,$$

where

$$v = E_{\mathcal{U}}^* \Phi E_{\mathcal{E}} C(z) u + E_{\mathcal{U}}^* h_0, \quad k_0 = S_{\mathcal{U}}^* \Phi E_{\mathcal{E}} C(z) u + S_{\mathcal{U}}^* h_0.$$

Recall that $\mathcal{H} = E_{\mathcal{U}} \mathcal{U} \oplus \lambda \mathcal{H}_0$. Let J be the operator from $E_{\mathcal{U}} \mathcal{U} \oplus \lambda \mathcal{H}_0$ to $\mathcal{U} \oplus \mathcal{H}_0$ defined by $J(E_{\mathcal{U}} u \oplus \lambda h_0) = u \oplus h_0$. Obviously, J is unitary and its inverse is given by $J^{-1}(u \oplus h_0) = E_{\mathcal{U}} u \oplus \lambda h_0$.

It follows that relative to the direct sum decomposition $\mathcal{U} \oplus \mathcal{H}_0$ the operator $JG(z)J^{-1}$ is given by the following 2×2 operator matrix:

$$JG(z)J^{-1} = \begin{bmatrix} E_{\mathcal{U}}^* \Phi E_{\mathcal{E}} C(z) & E_{\mathcal{U}}^* \\ S_{\mathcal{U}}^* \Phi E_{\mathcal{E}} C(z) & S_{\mathcal{U}}^* \end{bmatrix}. \tag{3.10}$$

But then

$$J\big(I - zG(z)\big)J^{-1} = \begin{bmatrix} I - zE_{\mathcal{U}}^*\Phi E_{\mathcal{E}}C(z) & -zE_{\mathcal{U}}^* \\ -zS_{\mathcal{U}}^*\Phi E_{\mathcal{E}}C(z) & I - zS_{\mathcal{U}}^* \end{bmatrix},$$

and hence, using a Schur complement argument, we have

$$J\big(I - zG(z)\big)^{-1}J^{-1} = \begin{bmatrix} \Delta(z)^{-1} & * \\ * & * \end{bmatrix},$$

where

$$\begin{aligned} \Delta(z) &= I - zE_{\mathcal{U}}^*\Phi E_{\mathcal{E}}C(z) - (-zE_{\mathcal{U}}^*)(I - zS_{\mathcal{U}}^*)^{-1}(-zS_{\mathcal{U}}^*\Phi E_{\mathcal{E}}C(z)) \\ &= I - zE_{\mathcal{U}}^*\Phi E_{\mathcal{E}}C(z) + zE_{\mathcal{U}}^*(I - zS_{\mathcal{U}}^*)^{-1}(I - zS_{\mathcal{U}}^* - I)\Phi E_{\mathcal{E}}C(z) \\ &= I - zE_{\mathcal{U}}^*(I - zS_{\mathcal{U}}^*)^{-1}\Phi E_{\mathcal{E}}C(z) \\ &= I - z\Phi(z)C(z) = I - \Theta(z)C(z). \end{aligned}$$

We also know that $F(z)|_{\mathcal{F}} = 0$. Thus $F(z)J^{-1}$ admits the following representation

$$F(z)J^{-1} = \begin{bmatrix} F_1(z) & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{H}_0 \end{bmatrix} \to \mathcal{Y}. \tag{3.11}$$

But then we have

$$\begin{aligned} H(z) &= F(z)\big(I - zG(z)\big)^{-1}E_{\mathcal{U}} \\ &= F(z)J^{-1}J\big(I - zG(z)\big)^{-1}J^{-1}JE_{\mathcal{U}} \\ &= \begin{bmatrix} F_1(z) & 0 \end{bmatrix} \begin{bmatrix} \Delta(z)^{-1} & * \\ * & * \end{bmatrix} \begin{bmatrix} I_{\mathcal{U}} \\ 0 \end{bmatrix} \\ &= F_1(z)\Delta(z)^{-1} = F_1(z)\big(I - \Theta(z)C(z)\big)^{-1}. \end{aligned}$$

Let τ be the canonical embedding of $\mathcal{H}_0$ into $\mathcal{H}$, that is, τ is defined by $\tau h_0 = h_0$. From (3.10) and (3.11) it follows that

$$\begin{bmatrix} F(z) \\ G(z) \end{bmatrix} J^{-1} = \begin{bmatrix} F_1(z) & 0 \\ \Phi E_{\mathcal{E}}C(z) & \tau_{\mathcal{H}_0} \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{H}_0 \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{H} \end{bmatrix}. \tag{3.12}$$

Since $\operatorname{Im}\tau$ is perpendicular to $\Phi E_{\mathcal{E}}\mathcal{E}$ we see that for $h = E_{\mathcal{U}}u \oplus \lambda h_0$ we have

$$\|F(z)h\|^2 + \|G(z)h\|^2 = \|F_1(z)u\|^2 + \|\Phi E_{\mathcal{E}}C(z)u\|^2 + \|h_0\|^2$$

But $\|h\|^2 = \|u\|^2 + \|h_0\|^2$, and hence

$$\|h\|^2 - \big(\|F(z)h\|^2 + \|G(z)h\|^2\big) = \|u\|^2 - \big(\|F_1(z)u\|^2 + \|\Phi E_{\mathcal{E}}C(z)u\|^2\big).$$

Since multiplication by Φ and the map $E_{\mathcal{E}}$ are isometries, we conclude that

$$W = \begin{bmatrix} F \\ G \end{bmatrix} \in \mathbf{S}\left(\mathcal{U}, \begin{bmatrix} \mathcal{Y} \\ \mathcal{H} \end{bmatrix}\right) \Longleftrightarrow Z = \begin{bmatrix} F_1 \\ C \end{bmatrix} \in \mathbf{S}\left(\mathcal{U}, \begin{bmatrix} \mathcal{Y} \\ \mathcal{E} \end{bmatrix}\right).$$

We have now shown that the representations given by (3.7) and (3.8) imply those given by (0.9), with $Z \in \mathbf{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{E})$. The reverse implication is obtained by reversing the arguments. □

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Variants of Ando-Hiai Inequality

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Abstract. We discuss two variables version of the Ando-Hiai inequality: For $A,\ B > 0$ and $\alpha \in [0,\ 1]$, if $A \sharp_\alpha B \leq I$, then

$$A^r \sharp_{\frac{\alpha r}{\alpha r+(1-\alpha)s}} B^s \leq I \quad \text{for} \quad r, s \geq 1.$$

Here $\sharp_\alpha$ is the α-geometric mean in the sense of Kubo-Ando. In this context, the Furuta inequality is understood as the one-sided version (the case of $s = 1$): If $A \sharp_\alpha B \leq I$, then

$$A^r \sharp_{\frac{\alpha r}{\alpha r+1-\alpha}} B \leq I \quad \text{for} \quad r \geq 1.$$

As a consequence, the Furuta inequality has an alternative simple proof. In addition, we point out that the obtained inequality is understood as the case $t = 1$ in the grand Furuta inequality.

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1. Introduction

Let $A,\ B$ be positive operators on a Hilbert space. In [11], the α-geometric operator mean for $\alpha \in [0,\ 1]$ is defined as

$$A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$$

if $A > 0$, i.e., A is invertible. First of all, we cite the Ando-Hiai inequality for convenience and this one is equivalent to the main result of log majorization by Ando-Hiai [1].

Ando-Hiai inequality. *For $A, B > 0$,*

$$A \sharp_\alpha B \leq 1 \ \Rightarrow \ A^r \sharp_\alpha B^r \leq 1 \text{ for } r \geq 1. \tag{1.1}$$

Recently, we discussed in [4] some relations between the Ando-Hiai inequality and the following theorem:

Theorem A. *If $A \geq B > 0$ and $\alpha \in [0,\ 1]$, then*

$$A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I \tag{1.2}$$

holds for $p, r \geq 0$.

It is an operator order version of our characterization of the chaotic order that for $A, B > 0$, $\log A \geq \log B$ if and only if (2) in Theorem A holds for $p, r \geq 0$. By the way, the base of Theorem A is the Furuta inequality, see [2], [3], [5], [6], [7], [8], [9] and [10]:

Furuta inequality. *If $A \geq B > 0$, then for each $r \geq 0$*

$$(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \leq A^{\frac{p+r}{q}}$$

holds for $p \geq 0$ and $q \geq 1$ satisfying $(1+r)q \geq p+r$.

Its crucial point is the case $(1+r)q = p+r$. So we recognize the following inequality as (FI), which is expressed in terms of the α-geometric mean $\natural_\alpha$:

(FI) If $A \geq B > 0$, then

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq A \quad \textit{for} \quad p \geq 1 \textit{ and } r \geq 0. \tag{1.3}$$

Based on these facts, we proposed two variables version of the Ando-Hiai inequality in [3; Theorem 3]:

Theorem 1. *For $A,\ B > 0$ and $\alpha \in [0, 1]$, if $A \natural_\alpha B \leq I$, then*

$$A^r \natural_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} B^s \leq I \quad \textit{for} \quad r, s \geq 1.$$

It is obvious that the case $r = s$ in Theorem 1 is just the Ando-Hiai inequality. Moreover, as discussed in below, the case $s = 1$ is a natural variational expression of the Furuta inequality. Namely Theorem 1 is a common extension of them.

2. Equivalence

Now we consider two one-sided versions of Theorem 1:

Proposition 2. *For $A,\ B > 0$ and $\alpha \in [0,\ 1]$, if $A \natural_\alpha B \leq I$, then*

$$A^r \natural_{\frac{\alpha r}{\alpha r + 1 - \alpha}} B \leq I \quad \textit{for} \quad r \geq 1.$$

Proposition 3. *For $A,\ B > 0$ and $\alpha \in [0,\ 1]$, if $A \natural_\alpha B \leq I$, then*

$$A \natural_{\frac{\alpha}{\alpha + (1-\alpha)s}} B^s \leq I \quad \textit{for} \quad s \geq 1.$$

In the section, we investigate relations among them and Theorem 1.

Theorem 4.

(1) *Propositions 2 and 3 are equivalent.*
(2) *Theorem 1 follows from Propositions 2 and 3.*

Proof. (1) We first note the transposition formula $X \natural_\alpha Y = Y \natural_\beta X$ for $\beta = 1-\alpha$. Therefore Proposition 2 (for β) is rephrased as follows:

$$B \natural_\beta A \le I \quad \Rightarrow \quad B^s \natural_{\frac{\beta s}{\beta s+\alpha}} A \le I \quad \text{for} \quad s \ge 1.$$

Using the transposition formula again, it coincides with Proposition 3 because

$$1 - \frac{\beta s}{\beta s + \alpha} = \frac{\alpha}{\beta s + \alpha} = \frac{\alpha}{(1-\alpha)s + \alpha}.$$

(2) Suppose that $A \natural_\alpha B \le I$ and $r, s \ge 1$ are given. Then it follows from Proposition 2 that $A^r \natural_{\alpha_1} B \le I$ for $\alpha_1 = \frac{\alpha r}{\alpha r + 1 - \alpha}$. We next apply Proposition 3 to it, so that we have

$$1 \ge A^r \natural_{\frac{\alpha_1}{\alpha_1 + (1-\alpha_1)s}} B^s = A^r \natural_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} B^s,$$

as desired.

3. Furuta inequality of Ando-Hiai type

First of all, we point out that Proposition 2 is an Ando-Hiai type reformulation of (FI):

Theorem 5. *Proposition* 2 *is equivalent to the Furuta inequality.*

Proof. For a given $p \ge 1$, we put $\alpha = \frac{1}{p}$. Then $A \ge B (\ge 0)$ if and only if

$$A^{-1} \natural_\alpha B_1 \le 1 \text{ for } B_1 = A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}}. \tag{3.1}$$

If $A \ge B > 0$, then (3.1) holds for $A,\ B > 0$, so that Proposition 2 implies that for any $r \ge 0$

$$1 \ge A^{-(r+1)} \natural_{\frac{\frac{r+1}{p}}{(1-\frac{1}{p}) + \frac{r+1}{p}}} B_1 = A^{-(r+1)} \natural_{\frac{1+r}{p+r}} B_1 = A^{-(r+1)} \natural_{\frac{1+r}{p+r}} A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}}.$$

Hence we have (FI);

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \le A.$$

Conversely suppose that (FI) is assumed. If $A^{-1} \natural_\alpha B_1 \le 1$, then $A \ge (A^{\frac{1}{2}} B_1 A^{\frac{1}{2}})^\alpha = B$, where $p = \frac{1}{\alpha}$. So (FI) implies that for $r_1 = r - 1 \ge 0$

$$A \ge A^{-r_1} \natural_{\frac{1+r_1}{p+r_1}} B^p = A^{-(r-1)} \natural_{\frac{r}{p+r-1}} A^{\frac{1}{2}} B_1 A^{\frac{1}{2}}.$$

Since $\frac{r}{p+r-1} = \frac{\alpha r}{1+\alpha r - \alpha}$, we have Proposition 2.

As in the discussion of the preceding section, Theorem 1 can be proved by showing Proposition 2. Finally we cite a proof of Proposition 2 for completeness. Since it is equivalent to the Furuta inequality, we have an alternative proof of it. It is done by the usual induction, and its technical point is a multiplicative property of the index $\frac{\alpha r}{(1-\alpha)+\alpha r}$ of $\natural$ as appeared below.

Proof of Proposition 2. For convenience, we show that if $A^{-1} \natural_\alpha B \le I$, then

$$A^{-r} \natural_{\frac{\alpha r}{(1-\alpha)+\alpha r}} B \le I \quad \text{for} \quad r \ge 1. \tag{3.2}$$

Now the assumption says that

$$C^\alpha = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^\alpha \le A.$$

For any $\epsilon \in (0,1]$, we have $C^{\alpha\epsilon} \le A^\epsilon$ by the Löwner-Heinz inequality and so

$$\begin{aligned} A^{-(1+\epsilon)} \natural_{\frac{\alpha(1+\epsilon)}{(1-\alpha)+\alpha(1+\epsilon)}} B &= A^{-\frac{1}{2}}(A^{-\epsilon} \natural_{\frac{\alpha(1+\epsilon)}{1+\alpha\epsilon}} A^{\frac{1}{2}} B A^{\frac{1}{2}}) A^{-\frac{1}{2}} \\ \le \quad A^{-\frac{1}{2}}(C^{-\alpha\epsilon} \natural_{\frac{\alpha(1+\epsilon)}{1+\alpha\epsilon}} C) A^{-\frac{1}{2}} &= A^{-\frac{1}{2}} C^\alpha A^{-\frac{1}{2}} = A^{-1} \natural_\alpha B \le I. \end{aligned}$$

Hence we proved the conclusion (3.2) for $1 \le r \le 2$. So we next assume that (3.2) holds for $1 \le r \le 2^n$. Then the discussion of the first half ensures that

$$(A^{-r})^{r_1} \natural_{\frac{\alpha_1 r_1}{(1-\alpha_1)+\alpha_1 r_1}} B \le I \quad \text{for } 1 \le r_1 \le 2, \text{ where } \alpha_1 = \frac{\alpha r}{(1-\alpha)+\alpha r}.$$

Thus the multiplicative property of the index

$$\frac{\alpha_1 r_1}{(1-\alpha_1)+\alpha_1 r_1} = \frac{\alpha r r_1}{(1-\alpha)+\alpha r r_1}$$

shows that (3.2) holds for all $r \ge 1$.

4. Grand Furuta inequality

As is well known, the grand Furuta inequality (GFI) interpolates Ando-Hiai and Furuta inequalities. It is expressed as follows:
(GFI) If $A \ge B \ge 0$ with $A > 0$ and $t \in [0,1]$, then

$$[A^{\frac{r}{2}}(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \le A^{1-t+r}$$

for $r \ge t$ and $p, s \ge 1$. We note that

(GFI) for $t = 1,\ r = s \iff$ (AH)
(GFI) for $t = 0,\ s = 1 \iff$ (FI)

In this section, we point out that (GFI) for $t = 1$ includes both Ando-Hiai and Furuta inequalities.

Since Ando-Hiai inequality is just (GFI;$t = 1$) for $r = s$, it suffices to check that Furuta inequality is contained in (GFI;$t = 1$). As a matter of fact, it is just (GFI;$t = 1$) for $s = 1$.

Theorem 6. *Furuta inequality* (FI) *is equivalent to* (GFI) *for* $t = s = 1$.

Proof. We write down (GFI; $t = 1$) for $s = 1$: If $A \ge B > 0$, then

$$[A^{\frac{r}{2}}(A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}}) A^{\frac{r}{2}}]^{\frac{r}{p-1+r}} \le A^r$$

for $p, r \ge 1$, or equivalently,

$$A^{-(r-1)} \natural_{\frac{r}{p-1+r}} B^p \le A$$

for $p, r \geq 1$. Replacing $r-1$ by r_1, (GFI; $t=1$) for $s=1$ is rephrased as follows: If $A \geq B > 0$, then

$$A^{-r_1} \sharp_{\frac{1+r_1}{p+r_1}} B^p \leq A$$

for $p \geq 1$ and $r_1 \geq 0$, which is nothing but Furuta inequality.

Furthermore Theorem 1, generalized Ando-Hiai inequality, is understood as the case $t=1$ in (GFI):

Theorem 7. (GFI; $t=1$) *is equivalent to Theorem* 1.

Proof. (GFI; $t=1$) is written as

$$A \geq B > 0 \;\Rightarrow\; [A^{\frac{r}{2}}(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^sA^{\frac{r}{2}}]^{\frac{r}{(p-1)s+r}} \leq A^r \quad (p, r, s \geq 1).$$

We here put

$$\alpha = \frac{1}{p}, \quad B_1 = A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}}.$$

Then we have

$$A \geq B > 0 \iff A^{-1} \sharp_{\frac{1}{p}} A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}} \leq 1 \iff A^{-1} \sharp_\alpha B_1 \leq 1$$

and for each $p, r, s \geq 1$

$$\begin{aligned} &[A^{\frac{r}{2}}(A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^sA^{\frac{r}{2}}]^{\frac{r}{(p-1)s+r}} \leq A^r \\ \iff& A^{-r} \sharp_{\frac{r}{(p-1)s+r}} (A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^s \leq 1 \\ \iff& A^{-r} \sharp_{\frac{\alpha r}{\alpha r+(1-\alpha)s}} B_1^s \leq 1. \end{aligned}$$

This shows the statement of Theorem 1.

5. Concluding remarks

A further extension of Theorem 7 is given. We state the following result.

Theorem B. ([7]; Corollary 2.2) *For every A, $B \geq 0$ and $0 \leq \alpha \leq 1$,*

$$(A \sharp_\alpha B)^h \succ_{(log)} A^r \sharp_{\frac{h\alpha}{s}} B^s \quad \text{for } r \geq 1 \text{ and } s \geq 1,$$

where $h = [\alpha s^{-1} + (1-\alpha)r^{-1}]^{-1}$.

Theorem 8. (GFI; $t=1$) *is equivalent to Theorem* B.

It is easily verified that Theorem B is equivalent to (GFI; $t=1$) which is Corollary 1.3 itself in [7] and it is interesting to show that Theorem B is also equivalent to Theorem 1 since

$$\frac{h\alpha}{s} = \frac{\alpha}{[\alpha s^{-1} + (1-\alpha)r^{-1}]s} = \frac{\alpha r}{\alpha r + (1-\alpha)s}.$$

Finally we shall show a further extension of Theorem 8. We define $A \natural_s B$ for any $s \geq 0$ and for $A > 0$ and $B \geq 0$ by

$$A \natural_s B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^sA^{\frac{1}{2}}.$$

Theorem C. ([7]; Theorem 2.1) *For every $A > 0$, $B \geq 0$, $0 \leq \alpha \leq 1$ and each $t \in [0,1]$*

$$(A \sharp_\alpha B)^h \succ_{(log)} A^{1-t+r} \sharp_\beta (A^{1-t} \natural_s B)$$

holds for $s \geq 1$ and $r \geq t$, where $\beta = \frac{\alpha(1-t+r)}{(1-\alpha t)s+\alpha r}$ and $h = \frac{(1-t+r)s}{(1-\alpha t)s+\alpha r}$.

Theorem 9. (GFI) *itself is equivalent to Theorem* C.

We would like to omit the proof of Theorem 9 since it is included in the proof of Theorem 2.1 in [7].

Needless to say, when $t = 1$, Theorem 9 becomes Theorem 8.

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On Local Spectral Theory

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Abstract. The "single-valued extension property" suggests a "local" version of the point spectrum of an operator.

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Suppose $T : X \to Y$ is a linear operator: then its range and null space are the subspaces

(0.1) $$T(X) = \{Tx : x \in X\} \subseteq Y \ ; \ T^{-1}(0) = \{x \in X : Tx = 0 \in Y\} \ .$$

When $X = Y$ iteration leads to the *hyperrange* and the *hyperkernel*:

(0.2) $$T^{\infty}(X) = \bigcap_{n=1}^{\infty} T^n(X) \ ; \ T^{-\infty}(0) = \bigcup_{n=1}^{\infty} T^{-1}(0) \ .$$

When X is a normed space we have also the *transfinite range*:

1. Definition *If X is a complex normed space write*

(1.1) $$\Xi(X) = \bigcup_{k>0} \Xi_k(X) \text{ with } \Xi_k(X) = \{\xi \in X^{\mathbf{N}_*} : \forall n \in \mathbf{N}_* \ , \ \|\xi_n\| \le k^n \|\xi_0\|\} \ .$$

If $T \in B(X)$ is bounded and linear on X then the transfinite range of T is given by

(1.2) $$T^{\omega}(X) = \{\xi_0 : \xi \in \Xi(X) \ , \ \forall n \in \mathbf{N}_* \ , \ \xi_n = T(\xi_{n+1})\} \ ,$$

and the transfinite kernel is the subspace

(1.3) $$T^{-\omega}(0) = \{x \in X : \|T^n x\|^{1/n} \to 0\} \ .$$

The holomorphic range is the subspace

(1.4) $$\{\lim_{z \to 0} (T - zI) f(z) : f \in \mathrm{Holo}(0, X)\} \ ,$$

and the holomorphic kernel points are given by

(1.5) $$\{g(0) : (T - zI)g(z) \equiv 0 \text{ with } g \in \mathrm{Holo}(0, X)\} \ .$$

Here if $K \subseteq \mathbf{C}$ we write Holo(K, X) for the set of functions $f : D_f \to X$ which are defined and holomorphic on some open set $K \subseteq D_f \subseteq \mathbf{C}$, and identify a point $\lambda \in \mathbf{C}$ with the singleton set $K = \{\lambda\}$. In fact, for complete spaces, the transfinite range and the holomorphic range coincide:

2. Theorem *If $T : X \to X$ is bounded and linear on a Banach space then the holomorphic and the transfinite range points $T^\omega(X)$ coincide, and the holomorphic kernel points of T are given by the intersection*

$$T^{-1}(0) \cap T^\omega(X) \ . \tag{2.1}$$

Proof. If $y = \xi_0 \in T^\omega(X)$ as in (1.2) define

$$f(z) \equiv \xi_0 + \sum_{j=1}^{\infty} z^n \xi_n \text{ on } D_f = \{|z| < 1/k\} \ ; \tag{2.2}$$

conversely pass to the Taylor series of f. If (1.4) holds on $\{|z| < \rho\}$ then, with $h\rho < 1$,

$$f(z) = \xi_0 + \sum_{n=1}^{\infty} z^n \xi_n \text{ with } \xi_0 = x \ , \ \|\xi_n\| \leq \ell h^n \leq (h + \ell\|x\|)^n \|x\| \ .$$

It is clear that the holomorphic kernel points (1.5) are contained in the null space $T^{-1}(0)$, and that if $y = g(0)$ is in the set (1.5), then $y = \lim_{z\to 0}(T - zI)f(z) \in T^\omega(X)$ with

$$f(z) \equiv (\delta_0 g)(z) = \frac{g(z) - y}{z} \ (z \neq 0) \ , \ = g'(0) \ (z = 0) \ ; \tag{2.3}$$

conversely if $y = \lim_{z\to 0}(T - zI)f(z)$ as in (1.4) then $y = g(0)$ as in (1.5) with

$$g(z) \equiv y + zf(z) \tag{2.4}$$

□

The transfinite range is usually written $K(T)$, and known as the *coeur analytique* of T, while the transfinite kernel (1.3), written $H_0(T)$, is called the *quasinilpotent part* of T. The analogue of (1.2), with the whole sequence space $X^{\mathbf{N}_*}$ in place of the subset $\Xi(X)$, is known as the *coeur algébrique*, and is the largest linear subspace $M \subseteq X$ for which $TM = M$. The space $\Xi(X) = X \otimes \Xi$ can be thought of as the tensor product between the space X and the sequence space $\Xi = \Xi(\mathbf{C})$, although we do not claim that it is closed under addition. Notice that the coordinate mapping

$$0^\wedge : \xi \mapsto \xi_0 \ (X^{\mathbf{N}_*} \to X) \tag{2.5}$$

is "one one at zero" when restricted to $\Xi(X)$. On the whole space $X^{\mathbf{N}_*}$ distinguish operators

$$T \otimes s : X^{\mathbf{N}_*} \to X^{\mathbf{N}_*} \text{ induced by } T \in B(X) \ , \ s : \mathbf{C}^{\mathbf{N}_*} \to \mathbf{C}^{\mathbf{N}_*} \ ,$$

in particular when $s \in \{u, v\}$ is either the forward or the backward shift:

$$(ut)_{n+1} = t_n \ , \ (ut)_0 = 0 \ ; \ (vt)_n = t_{n+1} \ . \tag{2.6}$$

Thus

$$vu = 1 \neq uv \text{ } on \text{ } \mathbf{C}^{\mathbf{N}_*} \text{ .} \tag{2.7}$$

With this notation we have expressions for the transfinite range and kernel: if $T \in B(X)$ then

$$T^\omega(X) = 0^\wedge\big(\Xi(X)_\cap(I \otimes 1 - T \otimes v)^{-1}(0)\big) \tag{2.8}$$

and

$$T^{-1}(0)_\cap T^\omega(X) = 0^\wedge\big(\Xi(X)_\cap(I \otimes u - T \otimes 1)^{-1}(0)\big) \text{ .} \tag{2.9}$$

The transfinite range and kernel have perturbation properties similar to the hyperrange and hyperkernel ([4] Theorem 7.8.3):

3. Theorem *If $T \in B(X)$ then*

$$T^\omega(X) = T(T^\omega(X)) \text{ } and \text{ } T^{-\omega}(0) = T^{-1}T^{-\omega}(0) \text{ ,} \tag{3.1}$$

and if $S \in \mathrm{comm}(T)$ then

$$S(T^{-\omega}(0)) \subseteq T^{-\omega}(0) \text{ ; } S(T^\omega(X)) \subseteq T^\omega(X) \text{ .} \tag{3.2}$$

If in particular S is in the invertible commutant of T,

$$S \in \mathrm{comm}^{-1}(T) = \mathrm{comm}(T)_\cap B(X)^{-1} \text{ ,} \tag{3.3}$$

then

$$T^{-\omega}(0) \subseteq (T - S)^\omega(X) \text{ .} \tag{3.4}$$

Proof. Inclusion (3.1) and both of the hyperinvariances (3.2) are clear; for (3.4) argue

$$x \in T^{-\omega}(0) \text{ , } S \in \mathrm{comm}^{-1}(T) \implies x = (S - T)\sum_{n=0}^{\infty} S^{-n-1}T^n x$$

and perturb S by small scalars. □

As Mbekhta [11] has noticed, the familiar "separation of spectra" associated with isolated points can be expressed in terms of the transfinite range and kernel:

4. Theorem *Suppose X is a Banach space, and $T \in B(X)$: then necessary and sufficient that*

$$0 \notin \mathrm{acc}\ \sigma(T) \tag{4.1}$$

is that both

$$T^\omega(X) + T^{-\omega}(0) = X \text{ } and \text{ } T^\omega(X)_\cap T^{-\omega}(0) = \{0\} \tag{4.2}$$

and

$$T^\omega(X) = \mathrm{cl}\ T^\omega(X) \text{ } and \text{ } T^{-\omega}(0) = \mathrm{cl}\ T^{-\omega}(0) \text{ .} \tag{4.3}$$

Proof. If $0 \in \mathbf{C}$ is (4.1) at worst an isolated point of spectrum then $T \in B(X)$ has a *(Koliha-)Drazin inverse*, $T^\times \in B(X)$ for which

$$T^\times = T^\times T T^\times \ , \ T T^\times = T^\times T \quad \textit{and} \qquad (4.4)$$
$$T + (I - T^\times T) \in B(X)^{-1} \ , \ \|T^n(I - T^\times T)\|^{1/n} \to 0 \ :$$

the projection $T^\bullet = T^\times T$ commutes with T and the restrictions of T to the range and null space of $T^\bullet$ are respectively invertible and quasinilpotent. It is rather easy to see that in this situation

$$T^\bullet(X) = T^\omega(X) \textit{ and } (I - T^\bullet)(X) = T^{-\omega}(0) \ : \qquad (4.5)$$

the range $T^\bullet(X) \subseteq T^\omega(X)$ and the subspace of quasinilpotency $(I - T^\bullet)(X) \subseteq T^{-\omega}(0)$, while the intersections $T^\bullet(X)_\cap T^{-\omega}(0) = \{0\} = (I - T^\bullet)(X)_\cap T^\omega(X)$. Thus both (4.2) and (4.3) hold. Conversely a Baire's theorem argument shows that if the transfinite kernel is closed then the restriction of the operator to it is quasinilpotent: notice

$$K_m = \{x \in X : m \le n \Longrightarrow \|T^n x\| \le m^{-n}\|x\|\} \Longrightarrow T^{-\omega}(0) \subseteq \bigcup_{m=1}^{\infty} K_m \ . \qquad (4.6)$$

The restriction of T to the transfinite range is now one-one, and by (3.1) onto. □

Schmoeger [15] has remarked that it is sufficient for (4.1) that (4.2) holds together with half of (4.3): it is sufficient that the transfinite range is closed, so that in the presence of (4.2) we can replace (4.3) by the weaker condition

$$T^\omega(X) = \mathrm{cl}\ T^\omega(X) \ . \qquad (4.7)$$

We shall describe an operator $T : X \to X$ as *hyperexact* if there is inclusion

$$T^{-1}(0) \subseteq T^\infty(X) \ : \qquad (4.8)$$

in an old condition of Saphar [14] the hyperrange is replaced by the *coeur algébrique*, or alternatively by the transfinite range. When in addition the operator has closed range

$$T(X) = \mathrm{cl}\ T(X) \qquad (4.9)$$

then these conditions all coincide, and in addition the transfinite range reduces to the hyperrange. The argument rests on the open mapping theorem, and [7] can be made to rely on "consorted range points":

5. Definition *If $T : X \to X$ is bounded and linear then $y \in T^\theta(X)$ is a "consorted range point" of T if there are (x_n) in X and (S_n) in $\mathrm{comm}^{-1}(T)$, and $x \in X$, for which*

$$(T - S_n)x_n = y \textit{ and } \|S_n\| + \|x_n - x\| \to 0 \ , \qquad (5.1)$$

and more generally $y \in T^{\sim,\theta}(X)$ a "consorted closure point" if

$$(T - S_n)x_n = y \textit{ and } \|S_n\| + \|S_n x_n\| \to 0 \ . \qquad (5.2)$$

We shall say that $y \in T^{-,\theta}(0)$ *is a "consorted kernel point" if there are* (y_n) *in* X *and* (S_n) *in* $\mathrm{comm}^{-1}(T)$ *for which*

$$(5.3) \qquad (T - S_n)y_n = 0 \text{ and } \|S_n\| + \|y_n - y\| \to 0 \ .$$

Like the holomorphic kernel points, the consorted kernel points are an intersection:

6. Theorem *There is inclusion*

$$(6.1) \qquad T^\omega(X) \subseteq T^\theta(X) \subseteq T^{\sim,\theta}(X) \ ,$$

and equality

$$(6.2) \qquad T^{-,\theta}(0) = T^{-1}(0)_\cap T^{\sim,\theta}(X) \ .$$

Proof. Inclusion (6.1) is clear. If (5.2) holds then

$$y_n = y + S_n x_n \Longrightarrow (T - S_n)y_n = Ty + S_n((T - S_n)x_n - y) \ ,$$

vanishing if also $Ty = 0$, and

$$\|S_n\| + \|y_n - y\| = \|S_n\| + \|S_n x_n\| \to 0 \ .$$

Conversely if (5.3) holds then certainly $Ty = 0$, and

$$x_n = S_n^{-1}(y_n - y) \Longrightarrow (T - S_n)x_n = S_n^{-1}(T - S_n)(y_n - y) = S_n^{-1}Ty + y \ ,$$

with again

$$\|S_n x_n\| = \|y_n - y\| \to 0 \qquad \square$$

If $T \in B(X)$ is onto, and hence open, then (Finch [3])

$$(6.3) \qquad X = T(X) = T^\omega(X) = T^\infty(X) \ :$$

this is because by the open mapping theorem there is $k > 0$ for which

$$(6.4) \qquad \forall y \in X \ , \ \exists y' \in X \ : y = Ty' \text{ and } \|y'\| \le k\|y\| \ .$$

Thus

$$(6.5) \qquad y \in X \Longrightarrow y = \xi_0 \in \Xi'(X) \subseteq \Xi(X) \ ,$$

where $\Xi'(X) = \bigcup_k \Xi'_k(X)$ with

$$(6.6) \qquad \Xi'_k(X) = \{\xi \in X^{\mathbf{N}_*} : \forall n \in \mathbf{N}_* \ , \ \|\xi_{n+1}\| \le k\|\xi_n\|\} \ .$$

For operators with closed range the relationship between hyperrange, holomorphic range and consorted range points is [7] delicate:

7. Theorem *There is inclusion*

$$(7.1) \qquad T^{-1}(0)_\cap T^\theta(X) \subseteq T^\infty(X) \ .$$

If T has closed range then

$$(7.2) \qquad T^{-1}(0) \subseteq T^\theta(X) \Longleftrightarrow T^{-1}(0) \subseteq T^\infty(X) \Longleftrightarrow T^{-1}(0) \subseteq T^\omega(X) \ ,$$

in which case

$$T^\theta(X) = T^\infty(X) = T^\omega(X) \ . \tag{7.3}$$

Proof. If (5.1) holds then certainly $y = Tx \in T(X)$, and if in addition $Ty = 0$ then for each $n, k \in \mathbf{N}$

$$x_n = TS_n^{-1}x_n = T^kS_n^{-k}x_n \in T^\infty(X) \ .$$

The rest of the argument is Theorem 2 and Theorem 3 of [7], noting that ([7] Theorem 1) the hyperexactness (4.8) transmits the range closedness (4.9) to all powers T^n of T, and hence to $T^\infty(X)$. □

An operator $T : X \to Y$ is *one-to-one* iff its null space is trivial:

$$T^{-1}(0) = \{0\} \ ; \tag{7.4}$$

when $Y = X$ the *point spectrum* $\pi^{left}(T)$ or set of "eigenvalues" of T consists of those $\lambda \in \mathbf{C}$ for which $T - \lambda I$ fails to be one to one. The "holomorphic kernel points" offer a local analogue:

8. Definition *We shall say that $T \in B(X)$ is holomorphically one-one if it has no nontrivial holomorphic kernel points,*

$$T^{-1}(0)_\cap T^\omega(X) = \{0\} \ , \tag{8.1}$$

and write

$$\pi^{\text{left}}_{\text{loc}}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \textit{ is not holomorphically one-one}\} \ . \tag{8.2}$$

Holomorphic kernel points correspond to local failure of the single-valued extension property:

9. Theorem *The holomorphic kernel points of $T \in B(X)$ are precisely the points $\lambda \in \mathbf{C}$ for which T does not have the single-valued extension property at λ, and there is inclusion*

$$\pi^{\text{left}}_{\text{loc}}(T) \subseteq \text{int } \pi^{\text{left}}_{\text{loc}}(T) \subseteq \pi^{\text{left}}(T) \ , \tag{9.1}$$

and also inclusion

$$\pi^{\text{left}}(T) \subseteq \pi^{\text{left}}_{\text{loc}}(T) \cup \tau^{\text{right}}(T) \ , \tag{9.2}$$

where τ^{right} is the "defect spectrum,"

$$\tau^{\text{right}}(T) = \{\lambda \in \mathbf{C} : (T - \lambda I)(X) \neq X\} \ . \tag{9.3}$$

Proof. Necessary and sufficient for 0 not to be a holomorphic kernel point is the implication

$$\xi = (\xi_0, \xi_1, \xi_2, \ldots) \in \Xi(X) \ , \ T(\xi) = 0 \implies \xi_0 = 0 \in X \ ; \tag{9.4}$$

necessary and sufficient for T not to have the single-valued extension property at 0 is the implication

$$\xi = (\xi_0, \xi_1, \xi_2, \ldots) \in \Xi(X) \ , \ T(\xi) = 0 \implies \xi = 0 \in \Xi(X) \ . \tag{9.5}$$

By the one-one-ness at zero (2.5) of the mapping $0^\wedge$ restricted to $\Xi(X)$, this identifies holomorphic one-one-ness with the single-valued extension property. For the inclusion (9.1) simply observe that if $Tx = 0$ with $0 \neq x = \xi_0 \in \Xi_k(X)$ then [5]

$$|\lambda| < k \Longrightarrow \lambda \in \pi^{\text{left}}(T) \ . \tag{9.6}$$

For (9.2) repeat the open mapping theorem argument (6.3). □

The closure of $\pi^{\text{left}}_{\text{loc}}(T)$ is known [10], [13] as the "analytic residuum" of T. From (9.1) and (9.2) it follows ([1] Corollary 4.36)

$$\sigma(T) = \pi^{\text{left}}_{\text{loc}}(T) \cup \tau^{\text{right}}(T) \ ; \tag{9.7}$$

also the boundary of the local eigenvalues is included in the defect spectrum. There is a spectral mapping theorem for the single-valued extension property:

10. Theorem *If $T \in B(X)$ and if p is a polynomial then the one way spectral mapping theorem holds,*

$$p\pi^{\text{left}}_{\text{loc}}(T) \subseteq \pi^{\text{left}}_{\text{loc}} p(T) \ , \tag{10.1}$$

with equality for non constant p.

Proof. This is given by Müller ([13] Theorem 14.6, Corollary 14.7), for functions holomorphic on, and not constant on any component of, some neighbourhood of the spectrum: cf. Aiena ([1] Theorem 2.39), Laursen and Neumann ([10] Theorem 3.36) and Colojoara and Foias ([3] Theorems 1.5, 1.6). For something potentially more elementary recall the argument for the point spectrum: if $T : X \to Y$ and $S : Y \to Z$ are arbitrary there is implication

$$S^{-1}(0) = \{0\} = T^{-1}(0) \Longrightarrow (ST)^{-1}(0) = \{0\} \Longrightarrow T^{-1}(0) = \{0\} \ . \tag{10.2}$$

With $Z = Y = X$ and $ST = TS$ we look for the analogous

$$\begin{gathered} S^{-1}(0) \cap S^\omega(X) = \{0\} = T^{-1}(0) \cap T^\omega(X) \\ \Longleftrightarrow (ST)^{-1}(0) \cap (ST)^\omega(X) = \{0\} \end{gathered} \ . \tag{10.3}$$

While (10.2) gives part of this, the inclusion for the transfinite range is the other way round. For polynomials [8], [13] it would be sufficient that (10.3) hold in two situations: when

$$(S, T) = (R^m, R^n) \text{ with } R \in B(X) \ , \tag{10.4}$$

and when

$$VS - TU = I \text{ with } \{U, V\} \subseteq \text{comm}(S, T) \ . \tag{10.5}$$

Backward implication in (10.3) is clear when (a special case of) (10.4) holds:

$$S = T^n \Longrightarrow T^{-1}(0) \subseteq (ST)^{-1}(0) \ , \ (ST)^\omega(X) = T^\omega(X) \tag{10.6}$$

□

If for example $S = T^2$ then backward implication in (10.3) follows from the factorization (with bounded linear $\mathbf{R}$)

$$(I \otimes u - T^2 \otimes 1)(I \otimes v^\sim) = \mathbf{R}(I \otimes u - T \otimes 1) \ , \tag{10.7}$$

where the modified backward shift $v^\sim$ is given by

(10.8) $$(v^\sim \xi)_n = \xi_{2n}\ ,\ n \in \mathbf{N}_*\ ,\ \xi \in \mathbf{C}^{\mathbf{N}_*}\ .$$

In matrix form

$$\begin{bmatrix} -T^2 & 0 & 0 & 0 & 0\dots \\ I & 0 & -T^2 & 0 & 0\dots \\ 0 & 0 & I & 0 & -T^2\dots \\ \dots & \dots & \dots & \dots & \dots\dots \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -T^2 & -T^3 & -T^4 & -T^5\dots \\ 0 & I & T & 0 & 0\dots \\ 0 & 0 & 0 & I & T\dots \\ \dots & \dots & \dots & \dots & \dots\dots \end{bmatrix} \begin{bmatrix} -T & 0 & 0 & 0 & 0\dots \\ I & -T & 0 & 0 & 0\dots \\ 0 & I & -T & 0 & 0\dots \\ \dots & \dots & \dots & \dots & \dots\dots \end{bmatrix}$$

Inspection of the arguments of [1], [3] and [10] shows that they rely on equality

(10.9) $$T^{-1}(0) \cap T^\omega(X) = T^{-1}(0) \cap G_X(T)$$

where

(10.10) $$G_X(T) = \bigcap_{\lambda \in \mathbf{C}} (T - \lambda I)^\omega(X) \subseteq \bigcap_{\lambda \in \mathbf{C}} (T - \lambda I)^\infty(X) = F_X(T)\ :$$

we discussed the larger space in [8]. A strengthened version of one-one-ness is boundedness below:

11. Definition *We shall declare that $T \in B(X)$ is holomorphically bounded below if there are positive constants $(h_\delta)_{\delta>0}$ for which there is implication, for arbitrary $\xi \in \Xi(X)$ and arbitrary $\delta > 0$,*

(11.1) $$\eta = (T \otimes 1 - I \otimes u)\xi \Longrightarrow \|\sum_{j=0}^\infty z^j \xi_j\| \le h_\delta \|\sum_{j=0}^\infty z^j \eta_j\| \text{ on } \{|z| \le \delta\}\ .$$

It follows that if $f_n \in \text{Holo}(0, X)$ for each $n \in \mathbf{N}$ then there is implication

(11.2) $$(T - zI)f_n(z) \to 0 \Longrightarrow f_n(z) \to 0\ :$$

this implies a localised version of what is known as "property (β)." It is not immediately clear what a "local defect spectrum" should be, but what is called the "local spectrum" is derived from the transfinite range: if $x \in X$ write

(11.3) $$\sigma_T(x) \equiv \tau_x^{\text{right}}(T) = \{\lambda \in \mathbf{C} : x \notin (T - \lambda I)^\omega(X)\}\ .$$

Our version of the notation is supported by equality

(11.4) $$\bigcup_{x \in X} \tau_x^{\text{right}}(T) = \tau^{\text{right}}(T)\ :$$

inclusion one way round is elementary, while the other uses the open mapping theorem. Evidently

(11.5) $$x = 0 \Longrightarrow \tau_x^{\text{right}}(T) = \emptyset\ ;$$

also in a Banach algebra A with identity 1

$$x = 1 \Longrightarrow \tau_x^{\text{right}}(L_a) = \sigma_A^{\text{right}}(a) \ , \ \tau_x^{\text{right}}(R_a) = \sigma_A^{\text{left}}(a) \ . \tag{11.6}$$

This is a tiny tip of the iceberg that is Allan's theorem [2].

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The Finite Sections Approach to the Index Formula for Band-dominated Operators

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Abstract. Recently J. Roe and two of the authors derived a formula which expresses the Fredholm index of a band-dominated operator on $l^p(\mathbb{Z})$ in terms of local indices of its limit operators. The proof makes thoroughly use of K-theory for C^*-algebras (which, of course, appears as a natural approach to index problems). The purpose of this short note is to develop a completely different approach to the index formula for band-dominated operators which is exclusively based on ideas and results from asymptotic numerical analysis.

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1. Introduction

Let $1 < p < \infty$. We work on the Banach space $l^p(\mathbb{Z})$ of all two-sided infinite sequences $x : \mathbb{Z} \to \mathbb{C}$ for which

$$\|x\|_p = \left(\sum_{n \in \mathbb{Z}} |x(n)|^p \right)^{1/p} < \infty.$$

A bounded linear operator A on $l^p(\mathbb{Z})$ is a *band operator* if its matrix representation $(a_{ij})_{i,\, j \in \mathbb{Z}}$ with respect to the standard basis of $l^p(\mathbb{Z})$ is a band matrix, i.e., if there is a $k \in \mathbb{N}$ such that $a_{ij} = 0$ whenever $|i - j| > k$. By a *band-dominated operator* we mean the norm limit of a sequence of band operators.

We are interested in Fredholm properties of band-dominated operators. Recall that a bounded linear operator A on a Banach space X is called a Fredholm

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operator if its kernel $\{x \in X : Ax = 0\}$ and its cokernel $X/\{Ax : x \in X\}$ are finite-dimensional linear spaces, and that the difference

$$\operatorname{ind} A := \dim \ker A - \dim \operatorname{coker} A$$

is called the Fredholm index of A.

The Fredholm properties of a band-dominated operator can be expressed in terms of its limit operators. To state these results we have to recall some more notations. For every $k \in \mathbb{Z}$, we consider the operator of shift by k,

$$U_k : l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \quad (U_k x)(m) := x(m-k).$$

Further, we let $\mathcal{H}$ stand for the set of all sequences $h : \mathbb{N} \to \mathbb{Z}$ which tend to infinity in the sense that given $C > 0$, there is an n_0 such that $|h(n)| > C$ for all $n \geq n_0$. An operator $A_h \in L(l^p(\mathbb{Z}))$ is called a *limit operator* of $A \in L(l^p(\mathbb{Z}))$ with respect to the sequence $h \in \mathcal{H}$ if the sequence $(U_{-h(n)} A U_{h(n)})_{n \in \mathbb{N}}$ tends strongly to A_h as $n \to \infty$ and if the adjoint sequence $(U_{-h(n)} A U_{h(n)})^* = (U_{-h(n)} A^* U_{h(n)})$ converges strongly as well. Notice that every operator can possess at most one limit operator with respect to a given sequence $h \in \mathcal{H}$. The set $\sigma_{op}(A)$ of all limit operators of a given operator A is called the *operator spectrum* of A.

The main result of [3] (see also [4] for a comprehensive account) reads as follows.

Theorem 1.1. *Let $A \in L(l^p(\mathbb{Z}))$ be a band-dominated operator. Then*

(a) *every sequence $h \in \mathcal{H}$ possesses a subsequence g such that the limit operator A_g of A with respect to g exists;*
(b) *the operator A is compact if and only if $\sigma_{op}(A) = \{0\}$;*
(c) *the operator A is Fredholm if and only if each of its limit operators is invertible and if the norms of their inverses are uniformly bounded;*
(d) *if A is a band operator, then A is Fredholm if and only if each of its limit operators is invertible.*

Next we recall the index formula for Fredholm band-dominated operators. Let P denote the projection operator

$$P : l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \quad (Px)(n) = \begin{cases} x(n) & \text{if} \quad n \geq 0 \\ 0 & \text{if} \quad n < 0, \end{cases}$$

the range of which will be identified with $l^p(\mathbb{Z}^+)$ in the natural way. The complementary projection $I - P$ is denoted by Q. Let A be a band-dominated operator on $l^p(\mathbb{Z})$. Then the operators PAQ and QAP are compact (they are of finite rank if A is a band operator), and so are the operators $A - (PAP + Q)(P + QAQ)$ and $A - (P + QAQ)(PAP + Q)$. Hence, a band-dominated operator A is Fredholm if and only if both $PAP + Q$ and $P + QAQ$ are Fredholm operators, in which case the Fredholm index of A is equal to the sum of the Fredholm indices of $PAP + Q$ and $P + QAQ$. We call $\operatorname{ind}_+(A) := \operatorname{ind}(PAP + Q)$ and $\operatorname{ind}_-(A) := \operatorname{ind}(P + QAQ)$ the *plus-index* and the *minus-index* of A. Thus,

$$\operatorname{ind} A = \operatorname{ind}_+(A) + \operatorname{ind}_-(A) \tag{1.1}$$

for every Fredholm band-dominated operator A. Notice also that the operator spectrum splits into

$$\sigma_{op}(A) = \sigma_+(A) \cup \sigma_-(A)$$

where $\sigma_+(A)$ and $\sigma_-(A)$ collect the limit operators of A which correspond to sequences tending to $+\infty$ and to $-\infty$, respectively.

Theorem 1.2. *Let $A \in L(l^p(\mathbb{Z}))$ be a Fredholm band-dominated operator. Then*

(a) *for all $B_+ \in \sigma_+(A)$ and $B_- \in \sigma_-(A)$,*

$$\text{ind}_+(B_+) = \text{ind}_+(A) \quad \textit{and} \quad \text{ind}_-(B_-) = \text{ind}_-(A);$$

(b) *all operators in $\sigma_+(A)$ have the same plus-index, and all operators in $\sigma_-(A)$ have the same minus-index;*

(c) *for arbitrarily chosen operators $B_+ \in \sigma_+(A)$ and $B_- \in \sigma_-(A)$,*

$$\text{ind}\, A = \text{ind}_+(B_+) + \text{ind}_-(B_-). \tag{1.2}$$

Of course, if A is Fredholm, then all operators in $\sigma_+(A)$ also have the same minus-index, which follows from (b), from the invertibility of all limit operators of A, and from (1.1). It is also evident that assertions (b) and (c) are immediate consequences of assertion (a).

Theorem 1.2 was proved in [2] in case $p = 2$, and its generalization to $p \in (1, \infty)$ was done in [6]. A natural approach to the computation of the Fredholm index is via K-theory for C^*-algebras. Indeed, the proof of Theorem 1.2 given in [2] is heavily based upon calculations of K-groups of C^*-algebras of band-dominated operators and of their related ideals.

The purpose of this note is to present a completely different proof of the index result in Theorem 1.2 which is exclusively based on ideas and results from asymptotic numerical analysis, especially from the theory of the finite sections method.

2. The finite sections method

For $n \in \mathbb{N}$, consider the projection operators

$$P_n : l^p(\mathbb{Z}^+) \to l^p(\mathbb{Z}^+), \quad (P_n x)(m) := \begin{cases} x(m) & \text{if } 0 \le m < n \\ 0 & \text{otherwise.} \end{cases}$$

Let $A \in L(l^p(\mathbb{Z}^+))$. The operators P_nAP_n considered as acting on $\text{im}\, P_n$ are called the *finite sections* of A. The sequence $(P_nAP_n)_{n\in\mathbb{N}}$ of the finite sections of A is called *stable* if the operators $P_nAP_n : \text{im}\, P_n \to \text{im}\, P_n$ are invertible for all sufficiently large n and if the norms of their inverses are uniformly bounded. If the sequence (P_nAP_n) is stable, then the operator A is invertible, and the finite sections method applies to the solution of the equation $Au = f$ in the sense that there is an n_0 such that the equations $P_nAP_nu_n = P_nf$ possess unique solutions $u_n \in \text{im}\, P_n$ for each $n \ge n_0$ and for each right-hand side $f \in l^p(\mathbb{Z}^+)$ and that these solutions converge in the norm of $l^p(\mathbb{Z}^+)$ to the solution of the equation $Au = f$.

In case A is a band-dominated operator, the stability of the finite sections sequence (P_nAP_n) can be characterized in terms of limit operators of A. The following is the main result of [5]. It states not only a stability criterion for the full sequence (P_nAP_n) of the finite sections, but also for each of its subsequences. The notation $\sigma_{op,\eta}(A)$ refers to the set of all limit operators of A with respect to subsequences of a given strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$. Further we introduce the flip operator

$$J : l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \quad (Jx)(n) := x(-n-1).$$

Evidently, $JPJ = Q$.

Theorem 2.1. *Let $A \in L(l^p(\mathbb{Z}^+))$ be a band-dominated operator and $\eta : \mathbb{N} \to \mathbb{N}$ a strongly monotonically increasing sequence. Then the sequence $(P_{\eta(n)}AP_{\eta(n)})$ is stable if and only if the operator A and all operators*

$$JQA_hQJ \quad \text{with} \quad A_h \in \sigma_{op,\eta}(A)$$

are invertible on $l^p(\mathbb{Z}^+)$.

3. Compact and Fredholm sequences

In the following two sections, we let $p = 2$. This restriction is not really necessary; but we would like to refer to the results of [7] which were derived for $p = 2$. The case of general $p \in (1, \infty)$ will be discussed in Section 5. We will see there that the general result is indeed a simple consequence of the index formula for $p = 2$.

It is convenient to consider the sequence (P_nAP_n) of the finite sections of an operator $A \in l^2(\mathbb{Z}^+)$ as an element of a C^*-algebra, $\mathcal{F}$, which is defined as follows. The algebra $\mathcal{F}$ is the set of all bounded sequences (A_n) of operators $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$ provided with element-wise defined operations, an element-wise defined involution, and the norm $\|(A_n)\| := \sup \|A_n\|$. One easily checks that $\mathcal{F}$ is indeed a C^*-algebra and that the set $\mathcal{G}$ of all sequences in $\mathcal{F}$ which tend to zero in the operator norm forms a closed ideal of $\mathcal{F}$.

The relevance of the algebra $\mathcal{F}$ in our context becomes evident from the fact that a sequence $(A_n) \in \mathcal{F}$ is stable if and only if the coset $(A_n) + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{G}$. Moreover, the algebra $\mathcal{F}$ and its quotient algebra $\mathcal{F}/\mathcal{G}$ also provide a natural frame to formulate a Fredholm theory for sequences (A_n) of matrices. We are going to recall the basic notations from [1, 7].

A sequence (K_n) in $\mathcal{F}$ is a *sequence of rank one matrices* if every matrix K_n has range dimension less than or equal to one. The smallest closed ideal of $\mathcal{F}$ which contains all sequences of rank one matrices will be denoted by $\mathcal{K}$. The product of a sequence of rank one matrices with another sequence in $\mathcal{F}$ is again a sequence of rank one matrices. Hence, the set $\mathcal{K}_0$ of all finite sums of sequences of rank one matrices forms an (in general, non-closed) ideal of $\mathcal{F}$ the closure of which is just the ideal $\mathcal{K}$. Consequently, a sequence $(A_n) \in \mathcal{F}$ belongs to $\mathcal{K}$ if and only if, for every $\varepsilon > 0$, there is a sequence $(K_n) \in \mathcal{F}$ such that

$$\sup_n \|A_n - K_n\| < \varepsilon \quad \text{and} \quad \sup_n \dim \operatorname{im} K_n < \infty. \tag{3.1}$$

In particular, the ideal $\mathcal{K}$ encloses the ideal $\mathcal{G}$. We refer to the elements of $\mathcal{K}$ as *compact sequences*. The role of the ideal $\mathcal{K}$ in numerical analysis can be compared with the role of the ideal of the compact operators in operator theory.

An appropriate notion of the rank of a sequence in $\mathcal{F}$ can be introduced as follows. A sequence $\mathbf{A} \in \mathcal{F}$ is said to have *finite essential rank* if it is the sum of a sequence (G_n) in $\mathcal{G}$ and of a sequence (K_n) with $\sup_n \operatorname{rank} K_n < \infty$. If $\mathbf{A}$ is of finite essential rank, then there is a smallest integer $r \geq 0$ such that $\mathbf{A}$ can be written as $(G_n) + (K_n)$ with $(G_n) \in \mathcal{G}$ and $\sup_n \operatorname{rank} K_n \leq r$. We call this integer the *essential rank* of $\mathbf{A}$ and write $\operatorname{ess\,rank} \mathbf{A} = r$. If $\mathbf{A}$ is not of finite essential rank, then we set $\operatorname{ess\,rank} \mathbf{A} = \infty$. Thus, the sequences of essential rank 0 are just the sequences in $\mathcal{G}$. Clearly, the sequences of finite essential rank form an ideal of $\mathcal{F}$ which is dense in $\mathcal{K}$, and

$$\operatorname{ess\,rank}(\mathbf{A} + \mathbf{B}) \leq \operatorname{ess\,rank} \mathbf{A} + \operatorname{ess\,rank} \mathbf{B},$$
$$\operatorname{ess\,rank}(\mathbf{AB}) \leq \min\{\operatorname{ess\,rank} \mathbf{A}, \operatorname{ess\,rank} \mathbf{B}\}$$

for arbitrary sequences $\mathbf{A}, \mathbf{B} \in \mathcal{F}$. Several equivalent descriptions of compact sequences can be found in [7], Section 4.

In correspondence with the ideal $\mathcal{K}$, one introduces an appropriate class of Fredholm sequences by calling a sequence $(A_n) \in \mathcal{F}$ *Fredholm* if it is invertible modulo the ideal $\mathcal{K}$ of the compact sequences. It is not hard to show that a sequence $(A_n) \in \mathcal{F}$ is Fredholm in this sense if and only if there are sequences $(B_n) \in \mathcal{F}$ and $(J_n) \in \mathcal{K}$ with $\sup_n \operatorname{rank} J_n < \infty$ such that

$$B_n A_n = I_n + J_n \quad \text{for all } n \in \mathbb{N} \tag{3.2}$$

(see Theorem 5.1 in [7]).

If the sequence $\mathbf{A} = (A_n) \in \mathcal{F}$ is Fredholm, then the sequence $\mathbf{AA}^* = (A_n A_n^*)$ is Fredholm, too. The smallest non-negative integer k for which there exist a sequence $(B_n) \in \mathcal{F}$ as well as a sequence $(J_n) \in \mathcal{K}$ of essential rank k such that

$$B_n A_n^* A_n = I_n + J_n \quad \text{for all } n \in \mathbb{N}$$

is called the *α-number* $\alpha(\mathbf{A})$ of the Fredholm sequence $\mathbf{A}$.

The *index* of a Fredholm sequence $\mathbf{A} \in \mathcal{F}$ is the integer

$$\operatorname{ind}(\mathbf{A}) := \alpha(\mathbf{A}) - \alpha(\mathbf{A}^*).$$

Observe that, in the case at hand, the index of a Fredholm sequence is always zero, which is a consequence of the fact that the operators A_n act on finite-dimensional spaces. So the more interesting quantity associated with a Fredholm sequence seems to be its α-number. But we will see that, on the other hand, the vanishing of the index of a Fredholm sequence allows one to make use of the index as a conservation quantity. Some general properties and equivalent characterizations of Fredholm sequences can be found in Chapter 6 of [1] and Section 5 of [7].

It is not hard to see that the finite sections sequence (P_nAP_n) of a band-dominated operator A on $l^2(\mathbb{Z}^+)$ is Fredholm in the above sense if and only if A is a Fredholm operator (Theorem 5.7 in [7]). The adventure of computing the α-number of the sequence (P_nAP_n) was undertaken in [7], Theorem 5.8. Here is the result.

Theorem 3.1. *Let $A \in l^2(\mathbb{Z}^+)$ be a Fredholm band-dominated operator. Then the sequence (P_nAP_n) of the finite sections of A is a Fredholm sequence, and*

$$\alpha(P_nAP_n) = \dim\ker A + \max\{\dim\ker(QA_hQ + P) : A_h \in \sigma_+(A)\}. \tag{3.3}$$

4. The index formula for $p = 2$

We are now in a position to settle the finite sections proof of the index formula for band-dominated operators. Its main ingredient is a subsequence version of Theorem 3.1 which is in Theorem 4.1 below. Given a strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$, we consider the C^*-algebra $\mathcal{F}_\eta$ of all bounded sequences (A_n) of operators $A_n : \operatorname{im} P_{\eta(n)} \to \operatorname{im} P_{\eta(n)}$ and introduce the according ideals $\mathcal{G}_\eta$ of all zero sequences in $\mathcal{F}_\eta$ and $\mathcal{K}_\eta$ of all compact sequences in $\mathcal{F}_\eta$. Equivalently, the elements of $\mathcal{F}_\eta$, $\mathcal{G}_\eta$ and $\mathcal{F}_\eta$ are the restrictions of sequences in $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{K}$ onto the range of η. Again we call the sequences in $\mathcal{K}_\eta$ *compact*, whereas the sequences which are invertible modulo $\mathcal{K}_\eta$ are called *Fredholm*. The α-number of a Fredholm sequence in $\mathcal{F}_\eta$ is defined in complete analogy to the α-number of a Fredholm sequence in $\mathcal{F}$.

Theorem 4.1. *Let $A \in L(l^2(\mathbb{Z}^+))$ be a Fredholm band-dominated operator and $\eta : \mathbb{N} \to \mathbb{N}$ a strongly monotonically increasing sequence. Then $(P_{\eta(n)}AP_{\eta(n)})$ is a Fredholm sequence in $\mathcal{F}_\eta$, and*

$$\alpha(P_{\eta(n)}AP_{\eta(n)}) = \dim\ker A + \max\{\dim\ker(QA_hQ + P) : A_h \in \sigma_{op,\eta}(A)\}. \tag{4.1}$$

This result can be proved in complete analogy to its "full" version Theorem 3.1, with only evident modifications needed. We omit the tedious repetition of the arguments from [7].

A situation of particular interest arises if the sequence η in Theorem 4.1 is chosen such that the limit operator A_η exists. In this case, the operator spectrum $\sigma_{op,\eta}(A)$ consists of exactly one operator, namely A_η.

Corollary 4.2. *Let $A \in L(l^2(\mathbb{Z}^+))$ be a Fredholm band-dominated operator, and let $h : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence for which the limit operator A_h exists. Then $(P_{h(n)}AP_{h(n)})$ is a Fredholm sequence in $\mathcal{F}_h$, and*

$$\alpha(P_{h(n)}AP_{h(n)}) = \dim\ker A + \dim\ker(QA_hQ + P). \tag{4.2}$$

If $A \in L(l^2(\mathbb{Z}^+))$ is a Fredholm band-dominated operator, then its adjoint has the same property, and if the limit operator of A with respect to the sequence h

exists, then the limit operator $(A^*)_h$ exists as well, and $(A^*)_h = (A_h)^*$. Applying the preceding corollary to the sequence $(P_{h(n)}A^*P_{h(n)})$ yields

$$\begin{aligned}\alpha(P_{h(n)}A^*P_{h(n)}) &= \dim\ker A^* + \dim\ker(Q(A^*)_hQ + P)\\ &= \dim\ker A^* + \dim\ker(Q(A_h)^*Q + P).\end{aligned} \tag{4.3}$$

We have already mentioned that

$$\alpha(P_{h(n)}AP_{h(n)}) = \alpha(P_{h(n)}A^*P_{h(n)}).$$

Thus, subtracting (4.3) from (4.2) and taking into account that $\operatorname{ind} A = \dim\ker A -$ $\dim\ker A^*$ for every Fredholm operator, we obtain

$$\operatorname{ind} A = -\operatorname{ind}(QA_hQ + P) \tag{4.4}$$

for every Fredholm band-dominated operator A on $l^2(\mathbb{Z}^+)$ and every strongly monotonically sequence h for which the limit operator A_h exists. Notice that the "ind" on the left-hand side of (4.4) refers to the Fredholm index of an operator on $l^2(\mathbb{Z}^+)$, whereas the "ind" on the right-hand side stands for the index of an operator acting on $l^2(\mathbb{Z})$. Of course, the index of a Fredholm operator $A \in L(l^2(\mathbb{Z}^+))$ coincides with the index of the associated operator $PAP + Q$ considered as acting on $l^2(\mathbb{Z})$. Thus, (4.4) can be written in the more symmetric form

$$\operatorname{ind}(PAP + Q) = -\operatorname{ind}(QA_hQ + P) \tag{4.5}$$

which holds for every Fredholm band-dominated operator A on $l^2(\mathbb{Z}^+)$ and every limit operator A_h in $\sigma_+(A)$.

Let now A be a Fredholm band-dominated operator on $l^2(\mathbb{Z}^-)$. Then JAJ is a Fredholm band-dominated operator on $l^2(\mathbb{Z}^+)$, and $\operatorname{ind} A = \operatorname{ind} JAJ$. Moreover, since $U_{-n}J = JU_n$ for every integer n, one has $U_{-n}JAJU_n = JU_nAU_{-n}J$, whence it follows that the limit operator A_h exists if and only if the limit operator $(JAJ)_{-h}$ exists and that $JA_hJ = (JAJ)_{-h}$ in this case. Inserting the operator JAJ into (4.4) one thus concludes

$$\begin{aligned}\operatorname{ind} JAJ &= -\operatorname{ind}(Q(JAJ)_hQ + P)\\ &= -\operatorname{ind}(JPA_{-h}PJ + JQJ) = -\operatorname{ind}(PA_{-h}P + Q)\end{aligned}$$

for each sequence h tending to $+\infty$ for which the limit operator $(JAJ)_h$ (equivalently, the limit operator A_{-h}) exists. Writing this identity in a symmetric form we get

$$\operatorname{ind}(QAQ + P) = -\operatorname{ind}(PA_hP + Q) \tag{4.6}$$

which holds for every Fredholm band-dominated operator A on $l^2(\mathbb{Z}^-)$ and every limit operator A_h in $\sigma_-(A)$.

Let now A be a Fredholm band-dominated operator on $l^2(\mathbb{Z})$. Then PAP and QAQ are Fredholm band-dominated operators on $l^2(\mathbb{Z}^+)$ and $l^2(\mathbb{Z}^-)$, respectively, and from (4.5) and (4.6) one gets

$$\begin{aligned}\operatorname{ind} A &= \operatorname{ind}(PAP + Q) + \operatorname{ind}(QAQ + P)\\ &= -\operatorname{ind}(QA_hQ + P) - \operatorname{ind}(PA_gP + Q)\end{aligned} \tag{4.7}$$

which holds for arbitrary limit operators $A_h \in \sigma_+(A)$ and $A_g \in \sigma_-(A)$. Since the limit operators A_h and A_g are invertible if A is Fredholm, one has

$$\operatorname{ind}(QA_hQ+P) = -\operatorname{ind}(PA_hP+Q) \quad \text{and} \quad \operatorname{ind}(PA_gP+Q) = -\operatorname{ind}(QA_gQ+P),$$

which, together with (4.7), implies the assertion in case $p = 2$. □

5. The index formula for general $p \in (1, \infty)$

There are several ways to establish the index formula in case of a general $p \in (1, \infty)$. The simplest way is to apply a result of [6] which says that both the Fredholm property and the Fredholm index of a band-dominated operator A on $l^p(\mathbb{Z})$ are independent of $p \in (1, \infty)$. With this information it is easy to derive the general case from the case $p = 2$ (see [6] for details).

An alternative way is to create a Fredholm theory for matrix sequences acting on Banach spaces and to follow the above proof, but now starting with a general p from the beginning. This demanding program was pursued by Rogozhin and one of the authors in a series of papers [8, 9, 10]. The main obstacle is that the singular value decomposition of a matrix, which played a prominent role in the Hilbert space theory in [1, 7], is of less use for operators on l^p-spaces. In this more general setting, the singular values (which are purely algebraic quantities) have to be replaced by the approximation numbers (which depend on the underlying norm).

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A New Proof that Rudin's Module is not Finitely Generated

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Abstract. It is known that Rudin constructed an infinitely generated submodule in the Hardy space over the bidisk. In this paper, we give a new proof that Rudin's module is not finitely generated.

Mathematics Subject Classification (2000). Primary 47B38; Secondary 47B35.

Keywords. Inner functions and the Hardy space over the bidisk.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $H^2(z)$ denote the classical Hardy space over $\mathbb{D}$ with the variable z. The Hardy space over the bidisk $H^2 = H^2(\mathbb{D}^2)$ is the tensor product Hilbert space $H^2(z) \otimes H^2(w)$ with variables z and w. Under pointwise multiplication, H^2 becomes a Hilbert module over the polynomial ring $\mathbb{C}[z, w]$. A closed subspace $\mathcal{M}$ of H^2 is called a submodule if $\mathcal{M}$ is invariant under the module action. In other words, a submodule $\mathcal{M}$ is an invariant subspace under multiplication operators by z and w.

In the classical Hardy space theory, Beurling proved that every submodule is generated by an inner function. Beurling's theorem is one of the important theorems for the structure theory of a single operator acting on a Hilbert space. However, in the theory of the Hardy space over the bidisk, it is known that the structure of submodules is extremely complicated. For example, Rudin gave an example of a submodule which is not finitely generated in [2]. The purpose of this paper is to give a new proof of the following theorem:

Theorem 1 (Rudin [2]**).** *Let $\mathcal{M}$ be the submodule consisting of all functions in H^2 which have a zero of order greater than or equal to n at $(\alpha_n, 0) = (1 - n^{-3}, 0)$ for any positive integer n (this submodule is called Rudin's module). Then $\mathcal{M}$ is not finitely generated.*

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2. A new proof of Theorem 1

First, we are going to investigate the structure of Rudin's module as a Hilbert space (cf. [3, 4]). The following observation was obtained in private communication with Professor Izuchi (Niigata University). Setting

$$\mathcal{M}_n = (z-\alpha_n)^n H^2 + (z-\alpha_n)^{n-1} w H^2 + \cdots + (z-\alpha_n) w^{n-1} H^2 + w^n H^2,$$

then $\mathcal{M}_n$ is the submodule consisting of all functions in H^2 which have a zero of order greater than or equal to n at $(\alpha_n, 0)$. Since Rudin's module $\mathcal{M}$ is the intersection of all $\mathcal{M}_n$'s, we have $\mathcal{M} = \bigcap_{n=1}^{\infty} \mathcal{M}_n$. Setting $b_n = b_n(z) = (\alpha_n - z)/(1-\alpha_n z)$, then we have

$$\begin{aligned}
\mathcal{M}_n &= (z-\alpha_n)^n H^2 + (z-\alpha_n)^{n-1} w H^2 + \cdots + (z-\alpha_n) w^{n-1} H^2 + w^n H^2 \\
&= b_n^n H^2 + b_n^{n-1} w H^2 + \cdots + b_n w^{n-1} H^2 + w^n H^2 \\
&= b_n^n H^2(z) \oplus b_n^{n-1} w H^2(z) \oplus \cdots \oplus b_n w^{n-1} H^2(z) \oplus w^n H^2.
\end{aligned}$$

Here we note that $H^2 = H^2(z) \otimes H^2(w) = \sum_{j=0}^{\infty} \oplus w^j H^2(z)$. Then we have

$$\begin{array}{ccccccccc}
\mathcal{M}_1 & = & b_1 H^2(z) & \oplus & w H^2(z) & \oplus & w^2 H^2(z) & \oplus & \cdots \\
\mathcal{M}_2 & = & b_2^2 H^2(z) & \oplus & b_2 w H^2(z) & \oplus & w^2 H^2(z) & \oplus & \cdots \\
\mathcal{M}_3 & = & b_3^3 H^2(z) & \oplus & b_3^2 w H^2(z) & \oplus & b_3 w^2 H^2(z) & \oplus & \cdots \\
\vdots & & \vdots & & & \vdots & & \vdots & \\
\mathcal{M}_n & = & b_n^n H^2(z) & \oplus & b_n^{n-1} w H^2(z) & \oplus & b_n^{n-2} w^2 H^2(z) & \oplus & \cdots \\
\vdots & & \vdots & & & \vdots & & \vdots &
\end{array}.$$

Hence we have

$$\mathcal{M} = q_0(z) H^2(z) \oplus q_1(z) w H^2(z) \oplus q_2(z) w^2 H^2(z) \oplus \cdots = \sum_{j=0}^{\infty} \oplus q_j(z) w^j H^2(z),$$

where we set

$$\begin{cases} q_0(z) = \prod_{n=1}^{\infty} b_n^n(z), \\ q_j(z) = q_{j-1}(z) / \prod_{n=j}^{\infty} b_n(z) \quad (j \geq 1). \end{cases} \tag{1}$$

Next, in order to prove Theorem 1, we need a lemma.

Lemma 1. *Let $\mathcal{M}$ be Rudin's module. Then*

$$\mathcal{M} \ominus ((z-\alpha_n)\mathcal{M} + w\mathcal{M}) = \sum_{j=0}^{n} \oplus \mathbb{C} k_n(z) q_j(z) w^j,$$

where we set $k_n(z) = 1/(1-\alpha_n z)$.

Proof. It is easy to check that

$$\mathcal{M} \ominus (z-\alpha_n)\mathcal{M} = \sum_{j=0}^{\infty} \oplus \mathbb{C} k_n(z) q_j(z) w^j.$$

Let $f = \sum c_j k_n(z) q_j(z) w^j$ be a non-zero function in $\mathcal{M} \ominus ((z-\alpha_n)\mathcal{M} + w\mathcal{M})$. Then we have that

$$\begin{aligned}\left\langle \sum_{j=0}^{\infty} c_j k_n(z) q_j(z) w^j, q_k(z) w^{k+1} \right\rangle &= c_{k+1} \langle k_n(z) q_{k+1}(z), q_k(z) \rangle \\ &= c_{k+1} \overline{(q_k/q_{k+1})(\alpha_n)}.\end{aligned}$$

Since f is orthogonal to $w\mathcal{M}$, we have $c_{k+1}\overline{(q_k/q_{k+1})(\alpha_n)} = 0$ for each $k \geq 0$. Further, by (1), it follows that $(q_k/q_{k+1})(\alpha_n) \neq 0$ for any $k \geq n$. Hence we have $c_{k+1} = 0$ for any $k \geq n$, that is, we have that

$$\mathcal{M} \ominus ((z-\alpha_n)\mathcal{M} + w\mathcal{M}) \subset \sum_{j=0}^{n} \oplus \mathbb{C} k_n(z) q_j(z) w^j.$$

Conversely, it is easy to check that $k_n(z) q_j(z) w^j$ belongs to $\mathcal{M} \ominus w\mathcal{M}$ for every $0 \leq j \leq n$. This concludes the proof. □

A new proof of Theorem 1. Let $\operatorname{rank} \mathcal{M}$ denote the least cardinality of a generating set of $\mathcal{M}$. We note that the following formula given by Douglas and Yang in [1] is well known:

$$\dim \{\mathcal{M} \ominus ((z-\lambda)\mathcal{M} + (w-\mu)\mathcal{M})\} \leq \operatorname{rank} \mathcal{M},$$

where (λ, μ) is any point in $\mathbb{D}^2$. By Lemma 1, we have that

$$n+1 = \dim \{\mathcal{M} \ominus ((z-\alpha_n)\mathcal{M} + w\mathcal{M})\} \leq \operatorname{rank} \mathcal{M}$$

for every $n \geq 1$. Hence $\operatorname{rank} \mathcal{M}$ is not finite. This concludes the proof. □

Remark. Our proof can be applied to more general cases including Rudin's module. We will continue this research in a future work.

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Quasinilpotent Part of class A or (p, k)-quasihyponormal Operators

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Abstract. Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. T is called a class A operator if $|T|^2 \leq |T^2|$, and (p, k)-quasihyponormal if $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$. In this paper, we characterize the quasinilpotent part $\mathcal{H}_0(T - \lambda) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|(T - \lambda)^n x\|^{\frac{1}{n}} = 0\}$ of a class A or (p, k)-quasihyponormal operator T.

Mathematics Subject Classification (2000). 47B20.

Keywords. Class A operator, (p, k)-quasihyponormal operator, quasinilpotent part.

Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. T is called a class A operator if $|T|^2 \leq |T^2|$, and (p, k)-quasihyponormal if $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$ for $k \in \mathbb{N}$ and $0 < p \leq 1$. Let $\lambda \in \mathbb{C}$. The quasinilpotent part of $T - \lambda$ is defined as $\mathcal{H}_0(T - \lambda) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|(T - \lambda)^n x\|^{\frac{1}{n}} = 0\}$. In general, $\ker(T - \lambda) \subset \mathcal{H}_0(T - \lambda)$ and $\mathcal{H}_0(T - \lambda)$ is not closed. However it is known that if T is hyponormal, i.e., $TT^* \leq T^*T$, then $\mathcal{H}_0(T - \lambda) = \ker(T - \lambda) \subset \ker(T - \lambda)^*$.

In this paper, we characterize the quasinilpotent part of a class A or (p, k)-quasihyponormal operator. This is a generalization of the hyponormal operator case.

Class A operators were definded by T. Furuta, M. Ito and T. Yamazaki [4]. (There is no relation with class $A_{\aleph_0}$.) It is known that a class A operator has many interesting properties. A p-hyponormal operator T ($(TT^*)^p \leq (T^*T)^p$ for $0 < p \leq 1$) is of class A, and a class A operator T is paranormal, i.e., $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for $x \in \mathcal{H}$. In the following result, (1) is due to [3], (2) to [16], (3) to [15], and (4) to [13].

Lemma 1. *Let T be of class A. Then the following assertions hold:*

(1) *The operator T has Bishop's property (β), i.e., if $f_n(z)$ is an analytic function on D and $(T - z)f_n(z) \to 0$ uniformly on each compact subset of D, then $f_n(z) \to 0$ uniformly on each compact subset of D.*

(2) *The restriction* $T|_{\mathcal{M}}$ *to an invariant subspace* $\mathcal{M}$ *of* T *is also of class* A.
(3) *If* $\lambda \neq 0$ *is an isolated point of* $\sigma(T)$, *then the Riesz idempotent* E_λ *with respect to* λ *is self-adjoint and* $E_\lambda \mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$.
(4) *We have an inequality*

$$\||T^2| - |T|^2\| \leq \|T(1,1) - T(1,1)^*\| \leq \frac{1}{\pi}\text{meas } \sigma(T),$$

where $T = U|T|$ *is the polar decomposition of* T *and* $T(1,1) = |T|U|T|$. *Moreover, if* meas $\sigma(T) = 0$, *then* T *is normal.*

Theorem 2. *Let* T *be of class* A. *Then* $\mathcal{H}_0(T-\lambda) = \ker(T-\lambda)$ *for* $\lambda \in \mathbb{C}$.

Proof. Let $F \subset \mathbb{C}$ be a closed set. Define the global spectral subspace by

$$\mathcal{X}_T(F) = \{x \in \mathcal{H} | \ \exists \text{analytic } f(z) : (T-z)f(z) = x \text{ on } \mathbb{C} \setminus F\}.$$

It is known that $\mathcal{H}_0(T-\lambda) = \mathcal{X}_T(\{\lambda\})$ by Theorem 2.20 of [1]. As T has Bishop's property (β) by Lemma 1(1), $\mathcal{X}_T(F)$ is closed and $\sigma(T|_{\mathcal{X}_T(F)}) \subset F$ by Proposition 1.2.19 of [11]. Hence $\mathcal{H}_0(T-\lambda)$ is closed and $T|_{\mathcal{H}_0(T-\lambda)}$ is of class A by Lemma 1(2). Since $\sigma(T|_{\mathcal{H}_0(T-\lambda)}) \subset \{\lambda\}$, $T|_{\mathcal{H}_0(T-\lambda)}$ is normal by Lemma 1(4). If $\sigma(T|_{\mathcal{H}_0(T-\lambda)}) = \emptyset$, then $\mathcal{H}_0(T-\lambda) = \{0\}$ and $\ker(T-\lambda) = \{0\}$. If $\sigma(T|_{\mathcal{H}_0(T-\lambda)}) = \{\lambda\}$, then $T|_{\mathcal{H}_0(T-\lambda)} = \lambda$ and $\mathcal{H}_0(T-\lambda) \subset \ker(T-\lambda)$. □

Remark. If $\lambda \neq 0$, then $\mathcal{H}_0(T-\lambda) = \ker(T-\lambda) \subset \ker(T-\lambda)^*$. Moreover, if $\lambda \in \sigma(T) \setminus \{0\}$ is an isolated point then $\mathcal{H}_0(T-\lambda) = \ker(T-\lambda) = \ker(T-\lambda)^*$. But if $\lambda = 0$, $\ker(T-\lambda) \subset \ker(T-\lambda)^*$ is not true. (See [18].)

Next we consider (p,k)-quasihyponormal operators. It is known that there exists a (p,k)-quasihyponormal operator which is not of class A, and there exists a class A operator which is not (p,k)-quasihyponormal. I.H. Kim proved many interesting properties of (p,k)-quasihyponormal operators (similar to p-hyponormal and quasihyponormal operators). For example, a Putnam type inequality, a Weyl type theorem and quasisimilarity for (p,k)-quasihyponormal operators ([8], [10]). In the following result, (1) is due to [8] and (2) to [15].

Lemma 3. *Let* T *be* (p,k)*-quasihyponormal. Then the following assertions hold:*

(1) *Assume* ran T^k *is not dense. Decompose*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad \mathcal{H} = [\text{ran } T^k] \oplus \ker T^{*k}$$

where [ran T^k] *is the closure of* ran T^k. *Then* T_1 *is p-hyponormal,* $T_3^k = 0$ *and* $\sigma(T) = \sigma(T_1) \cup \{0\}$.
(2) *The restriction* $T|_{\mathcal{M}}$ *to an invariant subspace* $\mathcal{M}$ *of* T *is also* (p,k)*-quasihyponormal.*

Lemma 4. *Let* T *be* (p,k)*-quasihyponormal. Then* T *has Bishop's property* (β).

Proof. Let $f_n(z)$ be analytic on D. Let $(T-z)f_n(z) \to 0$ uniformly on each compact subset of D. Then

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_{n1}(z) \\ f_{n2}(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_{n1}(z) + T_2 f_{n2}(z) \\ (T_3 - z)f_{n2}(z) \end{pmatrix} \to 0.$$

Since $T_3^k = 0$, T_3 has Bishop's property (β) and $f_{n2}(z) \to 0$. Hence $(T_1 - z)f_{n1}(z) \to 0$. An operator T_1 is p-hyponormal by Lemma 3, then T_1 is of class A and T has Bishop's property (β) by Lemma 1(1). Thus $f_{n1}(z) \to 0$. □

Lemma 5. *Let T be (p,k)-quasihyponormal. If $T|_{[\text{ran } T^k]}$ is normal, then* $[\text{ran } T^k]$ *reduces T.*

Proof. We may assume ran T^k is not dense. Decompose T into

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad H = [\text{ran } T^k] \oplus \ker T^{*k}$$

as in Lemma 3(1).

Let $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection of $\mathcal{H}$ onto $[\text{ran } T^k]$. Then

$$\begin{aligned} \begin{pmatrix} (T_1^*T_1)^p & 0 \\ 0 & 0 \end{pmatrix} &= (QT^*TQ)^p \geq Q(T^*T)^p Q \\ &\geq Q(TT^*)^p Q \geq Q(TQT^*)^p Q = \begin{pmatrix} (T_1T_1^*)^p & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

by Hansen's inequality [5] and Löwner-Heinz's inequality [6], [12]. Since $T_1 = T|_{[\text{ran } T^k]}$ is normal, we can write

$$(TT^*)^p = \begin{pmatrix} (T_1^*T_1)^p & A \\ A^* & B \end{pmatrix}.$$

Let $(TT^*)^{p/2} = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$. Then

$$\begin{aligned} \begin{pmatrix} (T_1^*T_1)^{p/2} & 0 \\ 0 & 0 \end{pmatrix} &= (Q(TT^*)^p Q)^{\frac{1}{2}} \geq Q(TT^*)^{\frac{p}{2}} Q \\ &= \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \geq Q(TQT^*)^{\frac{p}{2}} Q = \begin{pmatrix} (T_1^*T_1)^{\frac{p}{2}} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$X = (T_1^*T_1)^{\frac{p}{2}}.$$

On the other hand, a straightforward calculation shows

$$(TT^*)^p = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}^2 = \begin{pmatrix} X^2 + YY^* & XY + YZ \\ Y^*X + ZY^* & Y^*Y + Z^2 \end{pmatrix}.$$

Hence

$$(T_1^*T_1)^p = X^2 + YY^* = X^2.$$

This implies $Y = 0$ and

$$(TT^*)^{\frac{p}{2}} = \begin{pmatrix} (T_1^*T_1)^{\frac{p}{2}} & 0 \\ 0 & Z \end{pmatrix}.$$

Then

$$\begin{aligned} TT^* &= \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} T_1^* & 0 \\ T_2^* & T_3^* \end{pmatrix} \\ &= \begin{pmatrix} T_1T_1^* + T_2T_2^* & T_2T_3^* \\ T_3T_2^* & T_3T_3^* \end{pmatrix} = \begin{pmatrix} T_1^*T_1 & 0 \\ 0 & Z^{2/p} \end{pmatrix}. \end{aligned}$$

Thus $T_2T_2^* = 0$ and $T_2 = 0$. □

The following lemma is due to Kim [9].

Lemma 6. *Let T be (p, k)-quasihyponormal and $\mathcal{M}$ an invariant subspace of T. If $T|_{\mathcal{M}}$ is an injective normal operator, then $\mathcal{M}$ reduces T.*

Remark. The condition "$T|_{\mathcal{M}}$ is injective" is necessary. Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $T^2 = 0$. Hence T is $(p, 2)$-quasihyponormal and $T|_{\ker T} = 0$ is normal, but $\ker T$ does not reduce T.

Lemma 7. *Let $T \in B(\mathcal{H})$ be (p, k)-quasihyponormal and $\sigma(T) = \{\lambda\}$. Then $T = \lambda$ if $\lambda \neq 0$, and $T^k = 0$ if $\lambda = 0$.*

Proof. If the range of T^k is dense, then T is p-hyponormal operator. Hence $T = \lambda$ by Putnam's inequality for p-hyponormal operators [2]. If the range of T^k is not dense, then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = [\operatorname{ran} T^k] \oplus \ker T^{*k}$$

where T_1 is p-hyponormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$ by Lemma 3(1). In this case, $\lambda = 0$ and $T_1 = 0$ by [2]. Then $T_2 = 0$ by Lemma 5. Thus

$$T^k = \begin{pmatrix} 0 & 0 \\ 0 & T_3^k \end{pmatrix} = 0.$$ □

Theorem 8. *Let T be (p, k)-quasihyponormal. Then*

$$\mathcal{H}_0(T - \lambda) = \begin{cases} \ker(T - \lambda) & \text{if } \lambda \neq 0, \\ \ker T^k & \text{if } \lambda = 0. \end{cases}$$

Moreover, if $\lambda \neq 0$, then $\mathcal{H}_0(T - \lambda) = \ker(T - \lambda) \subset \ker(T - \lambda)^*$.

Proof. Since T has Bishop's property (β) by Lemma 4 and $\mathcal{H}_0(T - \lambda) = \mathcal{X}_T(\{\lambda\})$ by Theorem 2.20 of [1], $\mathcal{H}_0(T - \lambda)$ is closed and $\sigma(T|_{\mathcal{H}_0(T-\lambda)}) \subset \{\lambda\}$ by Proposition 1.2.19 of [11]. Let $S = T|_{\mathcal{H}_0(T-\lambda)}$. Then S is (p, k)-quasihyponormal by Lemma 3(2).

If $\sigma(S) = \sigma(T|_{\mathcal{H}_0(T-\lambda)}) = \emptyset$, then $\mathcal{H}_0(T - \lambda) = \{0\}$ and $\ker(T - \lambda) = \{0\}$.

If $\sigma(S) = \{\lambda\}$ and $\lambda \neq 0$, then $S = \lambda$ by Lemma 7 and $\mathcal{H}_0(T-\lambda) = \ker(S-\lambda) \subset \ker(T-\lambda)$.

If $\sigma(S) = \{0\}$, then $S = \lambda^k$ by Lemma 7 and $\mathcal{H}_0(T-\lambda) = \ker S^k \subset \ker T^k$.

Moreover, let $\lambda \neq 0$. In this case, $S = \lambda$. Hence S is normal and invertible, so $\mathcal{H}_0(T-\lambda)$ reduces T by Lemma 6. Thus $\mathcal{H}_0(T-\lambda) = \ker(T-\lambda) \subset \ker(T-\lambda)^*$. □

Remark. If $\lambda \neq 0$ is an isolated point of $\sigma(T)$, then $\mathcal{H}_0(T-\lambda) = \ker(T-\lambda) = \ker(T-\lambda)^*$. But if $\lambda = 0$, $\ker(T-\lambda) \subset \ker(T-\lambda)^*$ is not true. (See [15].)

Corollary 9. *Let T be algebraically (p,k)-quasihyponormal, i.e., there exists a non-constant polynomial $f(z)$ such that $f(T)$ is (p,k)-quasihyponormal. Then*

$$\mathcal{H}_0(T-\lambda) = \ker(T-\lambda)^{nk} \ \textit{where } n = \deg f$$

for $\lambda \in \mathbb{C}$.

Proof. Since $\ker(T-\lambda)^{nk} \subset \mathcal{H}_0(T-\lambda)$, we prove $\mathcal{H}_0(T-\lambda) \subset \ker(T-\lambda)^{nk}$. Since $f(T)$ has Bishop's property (β) by Lemma 4, T has Bishop's property (β) by Theorem 3.3.9 of [11]. Then $\mathcal{H}_0(T-\lambda) = \mathcal{X}_T(\{\lambda\})$ by Theorem 2.20 of [1] and $\mathcal{H}_0(T-\lambda)$ is closed by Proposition 1.2.19 of [11]. Let $S = T|_{\mathcal{H}_0(T-\lambda)}$. Then S is (p,k)-quasihyponormal by Lemma 3(2) and $\sigma(S) \subset \{\lambda\}$. Decompose

$$f(z) - f(\lambda) = a(z-\lambda)^m \Pi_{j=1}^{n-m}(z-\lambda_j)$$

where $1 \le m \le n$ and $\lambda \neq \lambda_j$. Let $x \in \mathcal{H}_0(T-\lambda)$. Then

$$\begin{aligned} x \in \mathcal{H}_0(S-\lambda) &= \mathcal{X}_S(\{\lambda\}) = \mathcal{X}_{f(S)}(\{f(\lambda)\}) \\ &= \begin{cases} \ker(f(S)-f(\lambda)) & \text{if } f(\lambda) \neq 0 \\ \ker f(S)^k & \text{if } f(\lambda) = 0 \end{cases} \end{aligned}$$

by Theorem 3.3.6 of [11] and Theorem 8. Thus

$$0 = (f(S)-f(\lambda))^k x = a(S-\lambda)^{mk} \Pi_{j=1}^{n-m}(S-\lambda_j)^k x,$$

and $(T-\lambda)^{nk} x = (S-\lambda)^{nk} x = 0$. □

Recently, I.H. Jeon and I.H. Kim [7] consider operators T satisfying $T^*|T|^2T \le T^*|T^2|T$, and showed several interesting properties of Riesz idempotent for $\lambda \in \sigma(T)$. In this paper, we prove the same results hold for a wider class of operators. Define T to be of quasi-class (A,k) if

$$T^{k*}(|T^2| - |T|^2)T^k \ge 0$$

for $k \in \mathbb{N}$.

Theorem 10. *Let T be of quasi-class (A,k). Assume that* ran T^k *is not dense, and decompose*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \textit{on} \quad \mathcal{H} = [\text{ran } T^k] \oplus \ker T^{*k}.$$

Then T_1 is of class A, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Let $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection of $\mathcal{H}$ onto $[\text{ran } T^k]$. Then

$$\begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = Q|T|^2Q \le Q|T^2|Q \le (Q|T^2|^2Q)^{1/2} = \begin{pmatrix} |T_1^2| & 0 \\ 0 & 0 \end{pmatrix}$$

by Hansen's inequality [5]. Hence T_1 is of class A. Let $x \in \ker T^{*k}$. Then

$$T^k x = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^k \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} * \\ T_3^k x \end{pmatrix} \in [\text{ran } T^k] \oplus \ker T^{*k}.$$

Hence $T_3^k = 0$. This implies $\sigma(T) = \sigma(T_1) \cup \{0\}$. □

Lemma 11. *Let T be of quasi-class (A, k). Then the following assertions hold:*

(1) *T has Bishop's property (β).*
(2) *The restriction $T|_{\mathcal{M}}$ to an invariant subspace $\mathcal{M}$ of T is also of quasi-class (A, k).*

Proof. The proof of (1) is the same as in Lemma 4. We prove (2). Decompose

$$T = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

Let $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. Then

$$|S|^2 = \left(Q|T|^2Q\right)|_{\mathcal{M}}$$

and

$$|S^2| = \left(Q|T^2|^2Q\right)^{1/2}|_{\mathcal{M}} \ge \left(Q|T^2|Q\right)|_{\mathcal{M}}.$$

Let $x \in \mathcal{M}$. Then

$$\begin{aligned} \langle S^{*k}|S|^2S^kx, x\rangle &= \langle S^{*k}(Q|T|^2Q)|_{\mathcal{M}}S^kx, x\rangle \\ &= \langle |T|^2T^kx, T^kx\rangle \le \langle |T^2|T^kx, T^kx\rangle \\ &= \langle S^{*k}(Q|T^2|Q)|_{\mathcal{M}}S^kx, x\rangle \le \langle S^{*k}|S^2|S^kx, x\rangle. \end{aligned}$$ □

Lemma 12. *Let $T \in B(\mathcal{H})$ be of quasi-class (A, k) and $\sigma(T) = \{\lambda\}$. Then $T = \lambda$ if $\lambda \ne 0$, and $T^{k+1} = 0$ if $\lambda = 0$.*

Proof. If the range of T^k is dense, then T is of class A. Hence $T = \lambda$ by Lemma 1(4). If the range of T^k is not dense, then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = [\text{ran } T^k] \oplus \ker T^{*k}$$

where T_1 is of class A, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$ by Theorem 10. In this case, $\lambda = 0$. Hence $T_1 = 0$ by Lemma 1(4). Thus

$$T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0.$$ □

Lemma 13. *Let T be of quasi-class (A,k). Then the following assertions hold:*

(1) *If $\mathcal{M}$ is an invariant subspace of T and $T|_{\mathcal{M}}$ is an injective normal operator, then $\mathcal{M}$ reduces T.*
(2) *If $(T-\lambda)x=0$ and $\lambda \neq 0$, then $(T-\lambda)^*x=0$.*

Proof. (1) Decompose T into

$$T = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$$

and let $S = T|_{\mathcal{M}}$ be an injective normal operator. Let Q be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. Since $T^k = \begin{pmatrix} S^k & * \\ 0 & B^k \end{pmatrix}$ and $\ker S = \ker S^* = \{0\}$, we have

$$\mathcal{M} = [\text{ran } S] = [\text{ran } S^k] \subset [\text{ran } T^k].$$

Then

$$\begin{aligned} \begin{pmatrix} |S|^2 & 0 \\ 0 & 0 \end{pmatrix} &= Q|T|^2Q \\ &\leq Q|T^2|Q \leq \left(Q|T^2|^2Q\right)^{1/2} = \begin{pmatrix} |S^2| & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

by Hansen's inequality [5]. Since S is normal, we can write

$$|T^2| = \begin{pmatrix} |S|^2 & C \\ C^* & D \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} |S|^4 & 0 \\ 0 & 0 \end{pmatrix} &= QT^*T^*TTQ \\ &= Q|T^2|\,|T^2|Q = \begin{pmatrix} |S|^4 + CC^* & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and $C = 0$. Hence

$$\begin{aligned} &\begin{pmatrix} |S|^4 & 0 \\ 0 & D^2 \end{pmatrix} = |T^2|^2 = T^*T^*TT \\ &= \begin{pmatrix} S^*S^*SS & S^*S^*(SA+AB) \\ (A^*S^* + B^*A^*)SS & (A^*S^* + B^*A^*)(SA+AB) + B^*B^*BB \end{pmatrix}. \end{aligned}$$

Since S is an injective normal operator, $SA + AB = 0$ and $D = |B^2|$.

If $k = 1$, then

$$\begin{aligned} 0 &\leq T^*(|T^2| - |T|^2)T \\ &= \begin{pmatrix} 0 & S^*S^*SA \\ A^*S^*SS & A^*S^*SA + B^*(|B^2| - |B|^2)B \end{pmatrix}. \end{aligned}$$

If $1 \leq k$, then

$$\begin{aligned} 0 &\leq T^{*k}(|T^2| - |T|^2)T^k \\ &= \begin{pmatrix} 0 & (-1)^{k+1}S^{*(k+1)}S^kA \\ (-1)^{k+1}A^*S^{*k}S^{k+1} & A^*S^{*k}S^kA + B^{*k}(|B^2| - |B|^2)B^k \end{pmatrix}. \end{aligned}$$

Thus $A = 0$.

(2) Let $\mathcal{M} = \text{span}\,\{x\}$. Then $T|_{\mathcal{M}} = \lambda \neq 0$ and $T|_{\mathcal{M}}$ is an injective normal operator. Hence $\mathcal{M}$ reduces T and $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Thus $(T - \lambda)^*x = 0$. □

Theorem 14. *Let T be of quasi-class (A, k). Then*

$$\mathcal{H}_0(T - \lambda) = \begin{cases} \ker(T - \lambda) & \text{if } \lambda \neq 0, \\ \ker T^{k+1} & \text{if } \lambda = 0. \end{cases}$$

Moreover, if $\lambda \neq 0$, then $\mathcal{H}_0(T - \lambda) = \ker(T - \lambda) \subset \ker(T - \lambda)^*$.

Proof. Since T has Bishop's property (β) by Lemma 11(1) and $\mathcal{H}_0(T - \lambda) = \mathcal{X}_T(\{\lambda\})$ by Theorem 2.20 of [1], $\mathcal{H}_0(T - \lambda)$ is closed and $\sigma(T|_{\mathcal{H}_0(T-\lambda)}) \subset \{\lambda\}$ by Proposition 1.2.19 of [11]. Let $S = T|_{\mathcal{H}_0(T-\lambda)}$. Then S is of quasi-class (A, k) by Lemma 11(2).

If $\sigma(S) = \sigma(T|_{\mathcal{H}_0(T-\lambda)}) = \emptyset$, then $\mathcal{H}_0(T - \lambda) = \{0\}$, and $\ker(T - \lambda) = \{0\}$.

If $\sigma(S) = \{\lambda\}$ and $\lambda \neq 0$, then $S = \lambda$ by Lemma 12, and $\mathcal{H}_0(T - \lambda) = \ker(S - \lambda) \subset \ker(T - \lambda)$.

If $\sigma(S) = \{0\}$, then $S^{k+1} = 0$ by Lemma 12, and $\mathcal{H}_0(T - \lambda) = \ker S^{k+1} \subset \ker T^{k+1}$.

Moreover, let $\lambda \neq 0$. In this case, $S = \lambda$. Hence S is normal and invertible, so $\mathcal{H}_0(T - \lambda)$ reduces T by Lemma 13(1). Thus $\mathcal{H}_0(T - \lambda) = \ker(T - \lambda) \subset \ker(T - \lambda)^*$. □

Next we consider an isolated point $\lambda \in \sigma(T)$. In this case, $E\mathcal{H} = \mathcal{H}_0(T - \lambda)$ where E denotes the Riesz idempotent for λ of T by Theorem 3.74 of [1].

Theorem 15. *Let T be of quasi-class (A, k). Let λ be an isolated point of $\sigma(T)$ and let E be the Riesz idempotent for λ. Then the following assertions hold.*

(1) *If $\lambda \neq 0$, then E is self-adjoint and*

$$E\mathcal{H} = \mathcal{H}_0(T - \lambda) = \ker(T - \lambda) = \ker(T - \lambda)^*.$$

(2) *If $\lambda = 0$, then $E\mathcal{H} = \mathcal{H}_0(T) = \ker(T^{k+1})$.*

Proof. (1) If the range $T^k\mathcal{H}$ is dense, then T is of class A. Hence we may assume $T^k\mathcal{H}$ is not dense by [18]. Then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = [T^k\mathcal{H}] \oplus \ker(T^{*k})$$

where T_1 is of class A, $\sigma(T) = \sigma(T_1) \cup \{0\}$ and $T_3^k = 0$ by Theorem 10. Hence $\lambda \in \sigma(T_1)$ is an isolated point of $\sigma(T_1)$. Then

$$\begin{aligned} E &= \frac{1}{2\pi i}\int_\gamma \begin{pmatrix} \mu - T_1 & -T_2 \\ 0 & \mu - T_3 \end{pmatrix}^{-1} d\mu \\ &= \frac{1}{2\pi i}\int_\gamma \begin{pmatrix} (\mu - T_1)^{-1} & (\mu - T_1)^{-1}T_2(\mu - T_3)^{-1} \\ 0 & (\mu - T_3)^{-1} \end{pmatrix} d\mu, \end{aligned}$$

where γ denotes the boundary of a closed disc $\Delta = \{\mu \in \mathbb{C} \ : \ |\mu - \lambda| \le r\}$ such that $0 \notin \Delta$ and $\Delta \cap \sigma(T) = \{\lambda\}$. Let

$$E_1 = \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1} d\mu$$

be the Riesz idempotent for λ of T_1. Then E_1 is selfadjoint and $E_1[T^k\mathcal{H}] = \ker(\lambda - T_1) = \ker(\lambda - T_1)^*$ by [18]. We prove that

$$X = \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}T_2(\mu - T_3)^{-1} d\mu = 0.$$

Since $(\mu - T_3)^{-1} = \dfrac{1}{\mu} + \dfrac{T_3}{\mu^2} + \dfrac{T_3^2}{\mu^3} + \cdots + \dfrac{T_3^{k-1}}{\mu^k}$, we see that

$$\begin{aligned} X &= \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}T_2\frac{1}{\mu} d\mu + \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}T_2\frac{T_3}{\mu^2} d\mu + \cdots \\ &= X_0 + X_1 + \cdots + X_{k-1}. \end{aligned}$$

Since $\dfrac{1}{\mu} = \dfrac{1}{\lambda} - \dfrac{\mu - \lambda}{\lambda^2} + \dfrac{(\mu - \lambda)^2}{\lambda^3} - \cdots$, we have

$$\begin{aligned} X_0 &= \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}T_2\frac{1}{\lambda} d\mu - \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}T_2\frac{\mu - \lambda}{\lambda^2} d\mu + \cdots \\ &= \frac{1}{\lambda}E_1T_2 - \frac{1}{\lambda^2}(T_1 - \lambda)E_1T_2 + \frac{1}{\lambda^3}(T_1 - \lambda)^2E_1T_2 - \cdots. \end{aligned}$$

We prove that

$$E_1T_2 = 0.$$

Let $x \in [T^k\mathcal{H}]$ and $y = E_1x$. Then $y \in \ker(\lambda - T_1) = \ker(\lambda - T_1)^* \subset \ker(T - \lambda)$. Hence

$$0 = (\lambda - T)^* \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} (\lambda - T_1)^* y \\ -T_2^* y \end{pmatrix}$$

by Lemma 13(2). Hence $T_2^*y = T_2^*E_1x = 0$. This implies $T_2^*E_1 = 0$ and $E_1T_2 = 0$. Hence $X_0 = 0$. Since $\frac{1}{\mu^2} = \frac{1}{\lambda^2} - \frac{2(\mu-\lambda)}{\lambda^3} + \frac{3(\mu-\lambda)^2}{\lambda^4} - \cdots$, we have

$$\begin{aligned}X_1 &= \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}T_2\frac{T_3}{\mu^2}d\mu \\ &= \frac{1}{\lambda^2}E_1T_2T_3 - \frac{2}{\lambda^3}(T_1-\lambda)E_1T_2T_3 + \frac{3}{\lambda^4}(T_1-\lambda)^2E_1T_2T_3 - \cdots \\ &= 0.\end{aligned}$$

Similarly we have $X_2 = X_3 = \cdots = X_{k-1} = 0$, and $X = 0$.

Hence $E = \begin{pmatrix} E_1 & 0 \\ 0 & 0\end{pmatrix}$ is selfadjoint and $E\mathcal{H} = \mathcal{H}_0(T-\lambda) = \ker(T-\lambda) \subset \ker(T-\lambda)^*$ by Theorem 14 and Lemma 13.

We prove $\ker(T-\lambda)^* \subset \ker(T-\lambda)$. Let $x = \begin{pmatrix} x_1 \\ x_2\end{pmatrix} \in \ker(T-\lambda)^*$. Then

$$0 = (T-\lambda)^*x = \begin{pmatrix} (T_1-\lambda)^*x_1 \\ T_2^*x_1 + (T_3-\lambda)^*x_2\end{pmatrix}.$$

Hence $x_1 \in \ker(T_1-\lambda)^* = \ker(T_1-\lambda) \subset \ker(T-\lambda)$. Then

$$0 = (T-\lambda)^*\begin{pmatrix} x_1 \\ 0\end{pmatrix} = \begin{pmatrix} (T_1-\lambda)^*x_1 \\ T_2^*x_1\end{pmatrix}$$

by Lemma 13, and $T_2^*x_1 = 0$. This implies $(T_3-\lambda)^*x_2 = 0$ and $x_2 = 0$ because T_3 is nilpotent. Thus $x = x_1 \in \ker(T_1-\lambda) \subset \ker(T-\lambda)$.

(2) The proof is straightforward from Theorem 14. □

Corollary 16. *Let T be algebraically of quasi-class (A,k), i.e., there exists a non-constant polynomial $f(z)$ such that $f(T)$ is of quasi-class (A,k). Then*

$$\mathcal{H}_0(T-\lambda) = \ker(T-\lambda)^{n(k+1)} \text{ where } n = \deg f$$

for $\lambda \in \mathbb{C}$.

Proof. Since $f(T)$ has Bishop's property (β) by Lemma 11(1), T has Bishop's property (β) by Theorem 3.3.9 of [11]. Then $\mathcal{H}_0(T-\lambda) = \mathcal{X}_T(\{\lambda\})$ by Theorem 2.20 of [1] and $\mathcal{H}_0(T-\lambda)$ is closed by Proposition 1.2.19 of [11]. Let $S = T|_{\mathcal{H}_0(T-\lambda)}$. Then S is of quasi-class (A,k) by Lemma 11(2) and $\sigma(S) \subset \{\lambda\}$. Decompose

$$f(z) - f(\lambda) = a(z-\lambda)^m\Pi_{j=1}^{n-m}(z-\lambda_j)$$

where $1 \le m \le n$ and $\lambda \ne \lambda_j$. Let $x \in \mathcal{H}_0(T-\lambda)$. Then

$$\begin{aligned}x \in \mathcal{H}_0(S-\lambda) &= \mathcal{X}_S(\{\lambda\}) = \mathcal{X}_{f(S)}(\{f(\lambda)\}) \\ &= \begin{cases}\ker(f(S)-f(\lambda)) & \text{if } f(\lambda) \ne 0 \\ \ker f(S)^{k+1} & \text{if } f(\lambda) = 0\end{cases}\end{aligned}$$

by Theorem 3.3.6 of [11] and Theorem 14. Thus

$$0 = (f(S) - f(\lambda))^{k+1}x = a(S - \lambda)^{m(k+1)}\Pi_{j=1}^{n-m}(S - \lambda_j)^{k+1}x,$$

and $(T - \lambda)^{n(k+1)}x = (S - \lambda)^{n(k+1)}x = 0$. □

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A New Majorization Induced by Matrix Order

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Abstract. Let h be a non-decreasing function on I and k an increasing function on J. Then h is said to be *majorized* by k, in symbols $h \preceq k$ if $J \subset I$ and $h \circ k^{-1}$ is operator monotone on $k(J)$, that is if $\sigma(A), \sigma(B) \subset J$, $k(A) \leqq k(B) \Longrightarrow h(A) \leqq h(B)$. We will give a condition for h in order that $h' \preceq h$.

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A and B stand for bounded selfadjoint operators on a Hilbert space. In this note it is assumed that a function is "real and continuous", and an increasing function is "strictly" increasing. A function $f(t)$ defined on an interval I in $\mathbf{R}$ is called an *operator monotone function* on I, if $A \leqq B$ implies $f(A) \leqq f(B)$ for every pair A and B whose spectra $\sigma(A)$ and $\sigma(B)$ lie in I. $\mathbb{P}(I)$ denotes the set of all operator monotone functions on I. When I is written in the concrete as $[a, b)$ we simply write $\mathbb{P}[a, b)$ instead of $\mathbb{P}([a, b))$. Suppose $I \subset J$. Then the restriction of $f \in \mathbb{P}(J)$ to I belongs to $\mathbb{P}(I)$. So we consider $\mathbb{P}(J)$ as the subset of $\mathbb{P}(I)$. We also consider as $\mathbb{P}_+(J) \subset \mathbb{P}_+(I)$; in particular, $\mathbb{P}_+[a, b) \subset \mathbb{P}_+(a, b)$. Since an operator monotone function is non-decreasing, $f \in \mathbb{P}_+(a, b)$ with $-\infty < a < \infty$ has the natural continuous extension to $[a, b)$, which belongs to $\mathbb{P}_+[a, b)$. This means $\mathbb{P}_+(a, b) \subset \mathbb{P}_+[a, b)$. Thus $\mathbb{P}_+(a, b) = \mathbb{P}_+[a, b)$. We remark that $\mathbb{P}[a, b) \subsetneqq \mathbb{P}(a, b)$; indeed, $1/(a - t) \in \mathbb{P}(a, b)$ but $1/(a - t) \notin \mathbb{P}[a, b)$. An increasing function $h(t)$ on I is called a *logarithmic operator monotone function* on I and denoted by $h \in \mathbb{LP}_+(I)$ if $h(t) > 0$ and $\log h$ is operator monotone on the interior of I. We put for $-\infty < a < b \leqq \infty$

$$\begin{aligned}\mathbb{P}_+^{-1}[a, b) \quad = \quad & \{h | h \text{ is increasing on } [a, b), \text{ its range is } [0, \infty), \text{ and} \\ & h^{-1} \in \mathbb{P}[0, \infty)\},\end{aligned}$$

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and for $-\infty \leqq a < b \leqq \infty$

$$\mathbb{P}_+^{-1}(a,b) = \{h | h \text{ is increasing on } (a,b), \text{ its range is } (0,\infty), \text{ and } h^{-1} \in \mathbb{P}(0,\infty)\}.$$

If $-\infty < a$, by identifying $h \in \mathbb{P}_+^{-1}(a,b)$ as its natural extension to $[a,b)$ we have

$$\mathbb{P}_+^{-1}(a,b) = \mathbb{P}_+^{-1}[a,b).$$

Let h be a non-decreasing function on I and k an increasing function on J. Then h is said to be *majorized* by k, in symbols

$$h \preceq k$$

if $J \subset I$ and $h \circ k^{-1}$ is operator monotone on $k(J)$. This definition is equivalent with

$$\sigma(A), \sigma(B) \subset J,\ k(A) \leqq k(B) \Longrightarrow h(A) \leqq h(B).$$

If we need to make clear the domain J of k, we write as follows:

$$h \preceq k \quad (J).$$

The following two lemmas shown in [9, 10] are essential. $h(a+0)$ stands for $\lim_{t \to a+0} h(t)$.

Lemma 1. (Product lemma) *Let I be a right open interval with end points a, b, where $-\infty \leqq a < b \leqq \infty$, and $h(t), g(t)$ non-negative functions defined on I such that the product hg is an increasing function with $(hg)(a+0) = 0$, $(hg)(b-0) = \infty$. Then for ψ_1, ψ_2 in $\mathbb{P}_+[0,\infty)$*

$$g \preceq hg \Longrightarrow h \preceq hg, \quad \psi_1(h)\psi_2(g) \preceq hg.$$

Lemma 2. (Closedness) *Let I be a right open interval with end points a, b, where $-\infty \leqq a < b \leqq \infty$. Suppose a sequence of functions h_n in $\mathbb{P}_+^{-1}(I)$ converges pointwise to h on I such that $h(a+0) = 0$, $0 < h(t)$ $(a < t)$ and $h(b-0) = \infty$. Then*

$$h \in \mathbb{P}_+^{-1}(I).$$

Moreover, suppose the sequence of increasing functions $\tilde{h}_n$ converges pointwise to a continuous function $\tilde{h}$ on I. Then

$$\tilde{h_n} \preceq h_n \ (n = 1, 2, \dots) \Longrightarrow \tilde{h} \preceq h.$$

In this lemma, we may assume more simply that h *is continuous and increasing with the range* $[0,\infty)$ instead of $h(a+0) = 0,\ 0 < h(t)\ (a < t), h(b-0) = \infty$. In [10], by using the above two lemmas we have shown

Theorem 3. (Product theorem) *For every right open interval I,*

$$\mathbb{P}_+^{-1}(I) \cdot \mathbb{P}_+^{-1}(I) \subset \mathbb{P}_+^{-1}(I), \quad \mathbb{LP}_+(I) \cdot \mathbb{P}_+^{-1}(I) \subset \mathbb{P}_+^{-1}(I).$$

Further, let $g_i(t) \in \mathbb{LP}_+(I)$ *for* $1 \leqq i \leqq m$ *and* $h_j(t) \in \mathbb{P}_+^{-1}(I)$ *for* $1 \leqq j \leqq n$*. Then for* $\psi_i, \phi_j \in \mathbb{P}_+[0,\infty)$

$$\prod_{i=1}^{m} \psi_i(g_i) \prod_{j=1}^{n} \phi_j(h_j) \preceq \prod_{i=1}^{m} g_i \prod_{j=1}^{n} h_j.$$

In [9] we showed $\mathbb{P}_+(I) \cdot \mathbb{P}_+^{-1}(I) \subset \mathbb{P}_+^{-1}(I)$ instead of the second inclusion in Product theorem. But it was futile in the case $I = (-\infty,\infty)$, for $\mathbb{P}_+(-\infty,\infty)$ consisits of constant functions. On the other hands, since $\mathbb{LP}_+(-\infty,\infty) = \{ce^{dt} | c > 0, d \geqq 0\}$ ([10]), the second inclusion in Product theorem is not meaningless; in fact, we will show, by using it, a generalized operator inequality of exponential type (see Theorem 5 below).

As an application we get

Theorem 4. ([9]) *For non-increasing sequences* $\{a_i\}_{i=1}^n$ *and* $\{b_i\}_{i=1}^m$*, define the positive and increasing functions* $u(t)$ *and* $v(t)$ *by*

$$u(t) = \prod_{i=1}^{n}(t-a_i) \quad (t \geqq a_1), \quad v(t) = \prod_{i=1}^{m}(t-b_i) \quad (t \geqq b_1).$$

Then $u(t) \in \mathbb{P}_+^{-1}[a_1,\infty)$*, and*

$$m \leqq n, \quad \sum_{i=1}^{k} b_i \leqq \sum_{i=1}^{k} a_i \ (1 \leqq k \leqq m) \Longrightarrow v \preceq u \quad ([a_1,\infty)).$$

This says that the "majorization between functions" defined above is an extension of the classical "submajorization between sequences".

Recall two well-known operator inequalities:
(Furuta inequality [4]): for $p \geqq 1, r > 0$

$$0 \leqq A \leqq B \Rightarrow \begin{cases} (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geqq (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}, \\ (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}} \leqq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}}. \end{cases} \tag{1}$$

(Ando [1], M. Fujii-T. Furuta-E. Kamei [3], also see [6]):
for $p, r > 0$

$$A \leqq B \Rightarrow \begin{cases} (e^{\frac{r}{2}A} e^{pB} e^{\frac{r}{2}A})^{\frac{r}{p+r}} \geqq (e^{\frac{r}{2}A} e^{pA} e^{\frac{r}{2}A})^{\frac{r}{p+r}}, \\ (e^{\frac{r}{2}B} e^{pA} e^{\frac{r}{2}B})^{\frac{r}{p+r}} \leqq (e^{\frac{r}{2}B} e^{pB} e^{\frac{r}{2}B})^{\frac{r}{p+r}}. \end{cases} \tag{2}$$

We remark that A and B in (1) are non-negative but those in (2) are not necessarily non-negative. The following is a simultaneous generalization of the above two inequalities:

Theorem 5. ([10]) *Let* I *be a right open interval,* $h(t) \in \mathbb{P}_+^{-1}(I)$*, and* $g(t) \in \mathbb{LP}_+(I)$*. Suppose* $\tilde{h}(t) \geqq 0$ *is a non-decreasing function on* I*. Then the function* φ *on* $(0,\infty)$ *defined by*

$$\varphi(g(t)h(t)) = g(t)\tilde{h}(t) \quad (t \in I)$$

belongs to $\mathbb{P}_+[0,\infty)$*, and for* A, B *with* $\sigma(A), \sigma(B) \subset I$

$$A \leqq B \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geqq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}, \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leqq g(B)^{\frac{1}{2}}\tilde{h}(A)g(B)^{\frac{1}{2}}. \end{cases}$$

Furthermore, if $\tilde{h} \in \mathbb{P}_+(I)$*, then*

$$A \leqq B \Rightarrow \begin{cases} \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \geqq \varphi(g(A)^{\frac{1}{2}}h(A)g(A)^{\frac{1}{2}}), \\ \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \leqq \varphi(g(B)^{\frac{1}{2}}h(B)g(B)^{\frac{1}{2}}). \end{cases} \tag{3}$$

(3) is a generalization of (1). In fact, consider $I = [0,\infty)$ and put $h(t) = t^p \in \mathbb{P}_+^{-1}(I)$, $g(t) = t^r \in \mathbb{LP}_+(I)$, and $\tilde{h}(t) = t \in \mathbb{P}_+(I)$ in (3), where $p \geqq 1$ and $r > 0$. Since $t \preceq t^p$ on I and the function φ on $[0,\infty)$ defined by $\varphi(t^p t^r) = tt^r$ is $t^{(1+r)/(p+r)}$, (3) yields (1). To see that it is also a generalization of (2), consider $I = (-\infty,\infty)$ and put $h(t) = e^{pt} \in \mathbb{P}_+^{-1}(I)$ for $p > 0$, $g(t) = e^{rt} \in \mathbb{LP}_+(I)$ for $r > 0$, and $\tilde{h}(t) = 1 \in \mathbb{P}_+(I)$. Since $1 \preceq e^{pt}$ on I and the function φ on $[0,\infty)$ defined by $\varphi(e^{pt}e^{rt}) = 1e^{rt}$ is $t^{r/(p+r)}$, (3) yields (2).

Thanks to the Löwner theorem, we can know a lot about f in $\mathbb{P}(I)$. Therefore we can know about h in $\mathbb{P}_+^{-1}(I)$ as well. So the next proposition says $\mathbb{P}_+^{-1}(I)$ is a good tool to study non-decreasing convex functions. We remark that the next proposition seems to contradict Lemma 2 at first sight.

Proposition 6. (cf. [5, 2]) *Let* $I = [a,\infty)$ *or* $I = (-\infty,\infty)$*. Suppose* $g(t)$ *is a non-negative convex function defined on* I *with* $g(a) = 0$ *or* $g(-\infty) = 0$ *respctively. Then there is a sequence* $\{h_n\}$ *so that* h_n *is a convex combination of functions in* $\mathbb{P}_+^{-1}(I)$ *and* $\{h_n\}$ *converges uniformly to* $g(t)$ *on every compact subset of* I*.*

Proof. Suppose $I = [a,\infty)$ and take c so that $a < c < \infty$. For every n consider the function $f_n(t)$ defined on $I = [a,\infty)$ by

$$f_n(t) = \frac{1}{2}\{\sqrt{(t-c)^2 + \frac{1}{n}} + t - a - \sqrt{(c-a)^2 + \frac{1}{n}}\}.$$

Since its inverse function

$$t + \frac{1}{2}(\sqrt{(c-a)^2 + \frac{1}{n}} + c + a) - \frac{1}{2n}\frac{1}{2t + \sqrt{(c-a)^2 + \frac{1}{n}} - c + a}$$

belongs to $\mathbb{P}_+[0,\infty)$, $f_n(t) \in \mathbb{P}_+^{-1}(I)$. As $n \to \infty$, $f_n(t)$ converges uniformly to $\frac{1}{2}\{|t-c| + t - c\}$. In the case where $I = (-\infty,\infty)$ put

$$f_n(t) = \frac{1}{2}\{\sqrt{(t-c)^2 + \frac{1}{n}} + t - c\}.$$

Then one can see $f_n(t) \in \mathbb{P}_+^{-1}(I)$ and $f_n(t)$ converges uniformly to $\frac{1}{2}\{|t-c|+t-c\}$ on I as $n \to \infty$. Suppose first $g = 0$. Then $\frac{1}{2}\{|t-n|+t-n\}$ converges uniformly to g on every compact subset of I as $n \to \infty$. Hence we can construct, by using $f_n(t)$ defined above, a sequence $\{h_n\}$ satisfying the required condition; for instance, put

$h_n(t) = \frac{1}{2}\{\sqrt{(t-n)^2 + \frac{1}{n}} + t - n\}$. Suppose next $g \neq 0$. Then from the assumption it follows that g is non-decreasing. Recall that g is continuous. Take a sequence $\{g_n(t)\}$ such that $g_n(t)$ is a finite convex combination of functions of the form $\frac{1}{2}\{|t-c| + t - c\}$ and $\{g_n(t)\}$ converges uniformly to $g(t)$ on every compact subset of I. Then one can easily construct a sequence $\{h_n\}$ which satisfies the required condition. □

From now on we study the relationship between a function and its derivative. It is evident that $-\frac{1}{t} \in \mathbb{P}(0,\infty)$ and $\int \frac{1}{t} dt = \log t \in \mathbb{P}(0,\infty)$. We have generalized this in Remark 1 of [8] as follows:

Lemma 7. *Let $f(t)$ be an increasing and differentiable function defined on (a,∞). If $-f'(t) \in \mathbb{P}(a,\infty)$, then $f(t) \in \mathbb{P}(a,\infty)$.*

The converse assertion of the above result does not hold; indeed, $f(t) := \frac{t}{t+1} \in \mathbb{P}(0,\infty)$, but $-f'(t) = \frac{-1}{(1+t)^2} \notin \mathbb{P}(0,\infty)$.

On the other hand, in Theorem 4 we have shown $u(t) \in \mathbb{P}_+^{-1}[a_1,\infty)$ and $u' \preceq u$. It is natural to ask when $f \in \mathbb{P}(I)$ implies $f' \in \mathbb{P}(I)$ and when $h' \preceq h$ holds for $h \in \mathbb{P}_+^{-1}(I)$. The following theorem clarifies them.

Theorem 8.

(i) $f(t) \in \mathbb{P}(0,\infty) \Rightarrow -f'(\sqrt{t}) \in \mathbb{P}(0,\infty)$.

(ii) $h(t) \in \mathbb{P}_+^{-1}[a,b) \Rightarrow h'(t) \preceq h(t)^2 \quad (a < t < b)$.

(iii) *Suppose $h \in \mathbb{P}_+^{-1}[a,b)$ and put $f = h^{-1}$. Then*

$$-f' \in \mathbb{P}(0,\infty) \Longleftrightarrow h' \preceq h.$$

Proof. (i). $f(t)$ is expressed as

$$f(t) = \alpha + \beta t + \int_0^\infty \left(-\frac{1}{t+s} + \frac{s}{s^2+1}\right) dv(s),$$

where α, β are real constants, $\beta \geqq 0$, and $\int_0^\infty \frac{1}{s^2+1} dv(s) < \infty$. For any $\delta > 0$ there is $M > 0$ such that $\frac{1}{(t+s)^2} \leqq M \frac{1}{1+s^2}$ for $\delta < t < \infty$, $0 < s < \infty$. Thus we have

$$f'(t) = \beta + \int_0^\infty \frac{1}{(t+s)^2} dv(s) \quad (\delta < t < \infty).$$

From this we get $-f'(\sqrt{t}) \in \mathbb{P}(\delta,\infty)$. Since δ is arbitrary, the desired result holds.

(ii). Put $f = h^{-1}$. Then $f \in \mathbb{P}[0,\infty)$ and hence $-f'(\sqrt{t}) \in \mathbb{P}(0,\infty)$ by (1). Since $h'(f(\sqrt{t})) = \frac{1}{f'(\sqrt{t})}$, we have $h'(f(\sqrt{t})) \in \mathbb{P}(0,\infty)$. This implies $h'(f(\sqrt{t})) \preceq t$ on $(0,\infty)$. The substitution of $h(t)^2$ for t in the above relation derives $h'(t) \preceq h(t)^2$ on $a < t < b$.

(iii). Since $f'(t) = \frac{1}{h'(f(t))}$, we have

$$-f'(t) \in \mathbb{P}(0,\infty) \Leftrightarrow h'(f(t)) \in \mathbb{P}(0,\infty) \Leftrightarrow h'(f(t)) \preceq t \Leftrightarrow h'(t) \preceq h(t).$$

The proof is completed. □

We remark that Theorem 8 gives us a way to get operator monotone functions. For instance, $\frac{t}{\log(1+t)} \in \mathbb{P}(0,\infty)$ derives

$$\frac{\sqrt{t}}{(1+\sqrt{t})\{\log(1+\sqrt{t})\}^2} - \frac{1}{\log(1+\sqrt{t})} \in \mathbb{P}(0,\infty)$$

from (1), and for $u(t)$ in Theorem 4, $u' \preceq u$ does $-(u^{-1})' \in \mathbb{P}(0,\infty)$ from (3) as well.

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H_∞ Functional Calculus and Characterization of Domains of Fractional Powers

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Abstract. Characterization of the domains of fractional powers of linear operators is very important in the study of (linear or nonlinear) abstract parabolic evolution equations. In this paper we present a new method of utilizing the H_∞ functional calculus of linear operators.

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Keywords. H_∞ functional calculus, domain of fractional power.

1. Introduction

Since McIntosh [43] invented the notion of bounded H_∞ functional calculus for sectorial operators with discovering equivalence with their various favorable properties presented by Yagi [55], the study of H_∞ functional calculus has attracted many mathematicians.

A sectorial operator A of a Banach space X is said to have a bounded H_∞ functional calculus if the functional calculus $f \mapsto f(A)$ is a bounded homomorphism from a Banach algebra $H_\infty(\Sigma)$ consisting of all bounded holomorphic function defined in a domain $\Sigma \subset \mathbb{C}$ into a Banach algebra $\mathcal{L}(X)$ of all bounded linear operators on X, where Σ is a sectorial domain containing the spectral set $\sigma(A)$ of A. Due to [43, 55], when A is a sectorial operator of a Hilbert space X, the boundedness of H_∞ functional calculus is equivalent to boundedness of imaginary powers A^{iy} $(y \in \mathbb{R})$ on X, to coincidence of the domain of fractional powers A^θ with the complex interpolation space $[X, \mathcal{D}(A)]_\theta$, $0 \leq \theta \leq 1$, and to integrability condition of the form

$$\int_\Gamma |\lambda|^{2\theta-1} |(A^{2(1-\theta)}(\lambda - A)^{-2} f, g)|\, |d\lambda| \leq C_\theta \|f\| \|g\|, \qquad f,\, g \in X;\ 0 < \theta < 1,$$

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Γ being a V-shaped integral contour edging the domain Σ. Such equivalence relations can partly be generalized for sectorial operators of reflexive Banach spaces as shown by Cowling-Doust-McIntosh-Yagi [8]. It is however known that these conditions are not all equivalent in general in Banach spaces. In fact, any bounded H_∞ functional calculus implies bounded imaginary powers, but the converse implication is not true.

Anyway, the theory of H_∞ functional calculus may be regarded as one of possible ways to develop the Fourier analysis for sectorial operators acting in Hilbert or Banach spaces. And the theory is actually utilized for establishing all kinds of the optimal properties for them, including the maximal regularity of solutions for abstract evolution equations, see Yagi [56], Dore-Venni [13], Giga-Giga-Sohr [25], Kalton-Weis [32], Prüss [47, 48], Arendt [4], Haase [29] and so forth.

When the underlying space X is a Hilbert space, we have some convenient sufficient conditions in order to know that a sectorial operator A has a bounded H_∞ functional calculus. Every self-adjoint operator has a bounded H_∞ functional calculus thanks to the spectral resolution. Kato [33, 34] proved that the maximal accretive operator A always has bounded imaginary powers $\|A^{iy}\| \le e^{\frac{\pi}{2}|y|}$ $(y \in \mathbb{R})$; consequently, the operator A has a bounded H_∞ functional calculus. More generally, if a sectorial operator A of a Hilbert space satisfies $\mathcal{D}(A^\theta) = (A^{*\theta})$ for an exponent $0 < \theta \le 1$, then the H_∞ functional calculus of A is bounded (see [55]). On the other hand, a counterexample of a sectorial operator which has no bounded H_∞ functional calculus was constructed by McIntosh-Yagi [45] in a Hilbert space. Afterwards, Baillon-Clément [5] constructed another counterexample in a different context.

When the space X is a Banach space, we have so far no convenient sufficient conditions (cf. [6]). We have to investigate each sectorial operator under consideration in a direct way. There are already a large number of literatures [2, 3, 10, 11, 14, 15, 16, 17, 18, 19, 20, 21, 31, 44, 52] giving positive results for many classes of elliptic differential operators equipped with suitable boundary operators acting in L_p spaces, where $1 < p < \infty$, $p \ne 2$. There are mainly two methods for establishing the boundedness of H_∞ functional calculus. The first one is obtaining a keen representation of the resolvent and applying the theory of pseudo-differential operators. The second one is of using integral kernels and estimating the kernels by employing the Calderón-Zygmund operator theory. The Stokes operator is handled by [12] and the Ornstein-Uhlenbeck operator is handled by [24, 42]. On the other hand, Hieber [30] presented a counterexample of pseudo-differential operator which has unbounded imaginary powers.

For a sectorial operator A of a Banach space X, A^z $(z = x + iy \in \mathbb{C})$ denote the fractional powers of A. As a matter of fact, the family A^z is a power function of A such that $A^0 = I$ (the identity mapping of X) and $A^1 = A$ which satisfies the Law of Exponent $A^{z+z'} = A^z A^{z'}$. For the precise definition and basic properties, we refer to Yosida [57], Komatsu [35, 36, 37, 38, 39, 40], Tanabe [53] and Carracedo-Alix [7].

This paper is, then, devoted to showing a new application of the theory of H_∞ functional calculus to the characterization problem of the domains of fractional powers of elliptic differential operators. Let A_p be a realization of an elliptic operator in L_p space under some boundary conditions. We want to characterize the domain $\mathcal{D}(A_p^\theta)$, $0 < \theta < 1$, in terms of the Sobolev-Lebesgue spaces. Such a characterization is known to be very important in the study of partial differential equations and, on the other hand, to be very difficult. There are only a few results on characterization. Fujiwara [22, 23] handled elliptic differential operators of second order, Seeley [49, 50] handled elliptic differential operators of general order, and Giga [26, 27] a realization of the Stokes operator in L_p. Their method of proof consists of two steps. Firstly, they prove boundedness of imaginary powers A_p^{iy} which implies coincidence of the domain $\mathcal{D}(A_p^\theta)$ with the interpolation space $[L_p, \mathcal{D}(A_p)]_\theta$ between the underlying L_p space and the domain $\mathcal{D}(A_p)$; secondly, they calculate precisely the interpolation space $[L_p, \mathcal{D}(A_p)]_\theta$.

In this paper we will present a direct method of using the H_∞ functional calculus for the characterization problem of $\mathcal{D}(A_p^\theta)$. We assume that A_p has a bounded H_∞ functional calculus. And, utilizing integrability condition of the form

$$\int_\Gamma |\lambda|^{2\theta-1} |\langle A_p^{2(1-\theta)}(\lambda - A_p)^{-2} f, g\rangle_{L_p \times L_{p'}}| \, |d\lambda| \\ \le C_\theta \|f\|_{L_p} \|g\|_{L_{p'}}, \qquad f \in L_p,\ g \in L_{p'};\ 0 < \theta < 1$$

which is actually equivalent to the boundedness of H_∞ functional calculus, we calculate directly the domain $\mathcal{D}(A_p^\theta)$ without interposing the interpolation space $[L_p, \mathcal{D}(A_p)]_\theta$. So, when A_2 is a maximal accretive operator in L_2 space (since A_2 always enjoys a bounded H_∞ functional calculus), we can calculate directly $\mathcal{D}(A_2^\theta)$ even in the case when the differential operator is considered in a non smooth domain $\Omega \subset \mathbb{R}^n$. When A_p is a sectorial operator in L_p space, where $1 < p < \infty$, $p \neq 2$, although we have to require the boundedness of H_∞ functional calculus which is stronger than the boundedness of imaginary powers A_p^{iy}, it is already known that elliptic operators of many classes enjoy this property. As a consequence, our direct method can be applicable for such many elliptic operators. We remark that our method has already been applied in author's previous paper [46] to the Laplace operator in L_2 space equipped with the Neumann boundary conditions in a convex polygonal domain. In this paper, we will apply to a second-order strongly elliptic operator A_p in L_p space equipped with the Dirichlet boundary conditions. Not only to this, our method is available even to elliptic operators of higher order which have a bounded H_∞ functional calculus.

The paper is organized as follows. In Section 2, we review the theory of H_∞ functional calculus build by [43] and [8]. Section 3 is devoted to preparing some known results in the theory of Sobolev-Lebesgue spaces. We consider a realization A_p of a second-order elliptic operator in the L_p space under the Dirichlet boundary conditions. The characterization results for $\mathcal{D}(A_2^\theta)$ and for $\mathcal{D}(A_p^\theta)$, $1 < p < \infty$, $p \neq 2$, are described in Sections 4 and 5, respectively.

2. H_∞ functional calculus

Let X be a Banach space with norm $\|\cdot\|$. Let A be a sectorial operator of X with angle $\omega_A < \pi$, namely, for any ω such that $\omega_A < \omega < \pi$, $\sigma(A) \subset \Sigma_\omega = \{\lambda \in \mathbb{C}; |\arg \lambda| < \omega\}$ and the estimate $\|(\lambda - A)^{-1}\| \leq M_\omega/|\lambda|$ holds for $\lambda \notin \Sigma_\omega$.

For $\omega_A < \omega \leq \pi$, let $H_\infty(\Sigma_\omega)$ denote the Banach algebra constituted of all bounded holomorphic functions defined in the sectorial domain Σ_ω equipped with the norm

$$\|f\|_\infty = \sup_{\lambda \in \Sigma_\omega} |f(\lambda)|, \qquad f \in H_\infty(\Sigma_\omega).$$

Let $\overset{\circ}{H}_\infty(\Sigma_\omega)$ be a subspace of $H_\infty(\Sigma_\omega)$ which is defined by

$$\overset{\circ}{H}_\infty(\Sigma_\omega) = \{f \in H_\infty(\Sigma_\omega); \ \sup_{\lambda \in \Sigma_\omega} |\lambda|^{\nu_f} |f(\lambda)| < \infty \ \text{with some } \nu_f > 0\}.$$

Then, for any function in $\overset{\circ}{H}_\infty(\Sigma_\omega)$, a functional calculus of A can be defined by the integral

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda, \qquad f \in \overset{\circ}{H}_\infty(\Sigma_\omega) \tag{2.1}$$

in the space $\mathcal{L}(X)$. Here, Γ is an integral contour such that

$$\begin{cases} \Gamma = \Gamma_- \cup \Gamma_0 \cup \Gamma_+, \\ \Gamma_0\colon \lambda = \delta e^{i\varphi} \quad (-\omega' \leq \varphi \leq \omega') \quad \text{and} \quad \Gamma_\pm\colon \lambda = \rho e^{\pm i\omega'} \quad (\delta \leq \rho < \infty), \end{cases} \tag{2.2}$$

ω' being any angle such that $\omega_A < \omega' < \omega$ and δ being any constant such that $0 < \delta < \|A^{-1}\|^{-1}$. Since $f(\lambda)(\lambda - A)^{-1}$ is integrable on Γ, $f(A)$ is a bounded operator of X and its definition is of course independent of the choice of ω' and δ.

Furthermore, for functions in $\overset{\circ}{H}_\infty(\Sigma_\omega)$, it is easily observed that

$$(f+g)(A) = f(A) + g(A) \quad \text{and} \quad (fg)(A) = f(A)g(A), \qquad f, g \in \overset{\circ}{H}_\infty(\Sigma_\omega).$$

If this correspondence $f \mapsto f(A)$ is extended from $H_\infty(\Sigma_\omega)$ to $\mathcal{L}(X)$ as a bounded homomorphism of Banach algebra, then A is said to have a bounded H_∞ functional calculus in Σ_ω. By the definition, if A has a bounded H_∞ functional calculus, then the estimate

$$\|f(A)\| \leq C_\omega \|f\|_\infty, \qquad f \in H_\infty(\Sigma_\omega) \tag{2.3}$$

must hold with some constant $C_\omega > 0$ as well as the formulas

$$(f+g)(A) = f(A)+g(A) \quad \text{and} \quad (fg)(A) = f(A)g(A), \qquad f, g \in H_\infty(\Sigma_\omega). \tag{2.4}$$

In view of (2.1) it is also immediate to observe that the correspondence $f \mapsto f(A)$ from $H_\infty(\Sigma_\omega)$ to $\mathcal{L}(X)$ is uniquely defined if A has a bounded H_∞ functional calculus. Similarly, the H_∞ functional calculus of A is seen to be characterized by the three conditions (2.1), (2.3) and (2.4).

Let us here introduce another important condition for the sectorial operator A. We say that A satisfies an integrable condition along the V-shaped contour

Γ_ω: $\lambda = \rho e^{\pm i\omega}$ $(0 \le \rho < \infty)$, where $\omega_A < \omega \le \pi$, with an exponent $0 < \theta < 1$, if the following estimate

$$\int_{\Gamma_\omega} |\lambda|^{2\theta-1} |\langle A^{2(1-\theta)}(\lambda - A)^{-2} f, g\rangle_{X\times X^*}|\, |d\lambda| \le C_{\omega,\theta}\|f\|\|g\|_*, \quad f \in X,\, g \in X^* \quad (2.5)$$

holds with some constant $C_{\omega,\theta} > 0$, where X^* is an adjoint space of X and where $\langle\cdot,\cdot\rangle_{X\times X^*}$ denotes the scaler product between X and X^*. When X is a Hilbert space, X^* is taken as X itself and $\langle\cdot,\cdot\rangle_{X\times X^*}$ is replaced by the inner product $(\cdot,\cdot)$ of X.

We then know the following equivalence theorem due to [43] and [55].

Theorem 2.1. *Let X be a Hilbert space. Let A be a sectorial operator of X with angle ω_A and let A^* be its dual operator. Then the following four conditions are mutually equivalent:*

1. *Along any V-shaped contour Γ_ω, $\omega_A < \omega \le \pi$, the integrable condition* (2.5) *holds for every exponent* $0 < \theta < 1$*;*
2. *In any domain Σ_ω, $\omega_A < \omega \le \pi$, A has a bounded H_∞ functional calculus;*
3. *The imaginary powers of A are bounded on X and satisfy estimates $\|A^{iy}\| \le N e^{\beta|y|}$ $(-\infty < y < \infty)$ with some exponent $\beta \ge 0$ and constant $N \ge 1$;*
4. *For every $0 < \theta < 1$, $\mathcal{D}(A^\theta) = [X, \mathcal{D}(A)]_\theta$ and $\mathcal{D}({A^*}^\theta) = [X, \mathcal{D}(A^*)]_\theta$ with norm equivalence.*

When X is a reflexive Banach space, this theorem is generalized in the following form as shown in [8].

Theorem 2.2. *Let X be a reflexive Banach space with adjoint space X^*. Let A be a sectorial operator of X with angle ω_A and let A^* be the adjoint operator of A. For A and A^*, consider the following four conditions:*

1. *Along any V-shaped contour Γ_ω, $\omega_A < \omega \le \pi$, the integrable condition* (2.5) *holds for every exponent* $0 < \theta < 1$*;*
2. *In any domain Σ_ω, $\omega_A < \omega \le \pi$, A has a bounded H_∞ functional calculus;*
3. *The imaginary powers of A are bounded on X and satisfy estimates $\|A^{iy}\| \le N e^{\beta|y|}$ $(-\infty < y < \infty)$ with some exponent $\beta \ge 0$ and constant $N \ge 1$;*
4. *For every $0 < \theta < 1$, $\mathcal{D}(A^\theta) = [X, \mathcal{D}(A)]_\theta$ and $\mathcal{D}({A^*}^\theta) = [X^*, \mathcal{D}(A^*)]_\theta$ with norm equivalence.*

Then, (1) *and* (2) *are mutually equivalent. Either of* (1) *or* (2) *implies* (3). (3) *implies* (4).

Consider a positive definite self-adjoint operator $A \ge \delta > 0$ in a Hilbert space X. Let $E(\lambda)$, $\delta < \lambda < \infty$ be its spectral resolution. Then, $A^{iy} = \int_\delta^\infty \lambda^{iy} dE(\lambda)$ and $\|A^{iy}\| \le 1$, $-\infty < y < \infty$. Therefore, A fulfills Condition (3) of Theorem 2.1.

Proposition 2.3. *The positive definite self-adjoint operator enjoys a bounded H_∞ functional calculus in any domain Σ_ω, $0 < \omega \le \pi$.*

A sectorial operator A of a Hilbert space is called a maximal accretive operator if it satisfies the condition $\mathrm{Re}\,(Au,u) \geq \delta\|u\|$ for $u \in \mathcal{D}(A)$ with some constant $\delta > 0$. The angle of a maximal operator is always smaller than $\frac{\pi}{2}$. According to [33, 34], any maximal accretive operator fulfills Condition (3) of Theorem 2.1 with $\|A^{iy}\| \leq e^{\frac{\pi}{2}|y|}$, $-\infty < y < \infty$. Therefore we can state the following assertion.

Proposition 2.4. *The maximal accretive operator enjoys a bounded H_∞ functional calculus in any domain Σ_ω, $\omega_A < \omega \leq \pi$, where $\omega_A \leq \frac{\pi}{2}$ is the angle of operator.*

Unfortunately, we do not know any convenient sufficient condition which ensures a bounded H_∞ functional calculus for the sectorial operator acting in a Banach space. As mentioned before, it is however known that realizations of a number of elliptic operators in the L_p space, $1 < p < \infty, p \neq 2$, under some boundary conditions actually enjoy Property (2) of Theorem 2.2.

3. Sobolev-Lebesgue spaces

Let Ω be a bounded domain of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. For $-\infty < s < \infty$ and $1 < p < \infty$, $H_p^s(\Omega)$ denotes the Sobolev-Lebesgue space in Ω. We describe a brief definition of $H^s(\Omega)$ and some basic properties which we need in this paper. For details, we refer to [1, 51, 54].

When $\Omega = \mathbb{R}^n$, the space is defined by

$$H_p^s(\mathbb{R}^n) = \{u \in \mathcal{S}(\mathbb{R}^n)'; \mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}u] \in L_p(\mathbb{R}^n)\}, \tag{3.1}$$

where $\mathcal{S}(\mathbb{R}^n)'$ denotes the set of tempered distributions in $\mathbb{R}^n$, and $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform on $\mathcal{S}(\mathbb{R}^n)'$, respectively. The space $H_p^s(\mathbb{R}^n)$ is a Banach space with the norm

$$\|u\|_{H_p^s} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}u]\|_{L_p}, \qquad u \in H_p^s(\mathbb{R}^n).$$

When Ω is a bounded domain with Lipschitz boundary and when $0 \leq s < \infty$, $H_p^s(\Omega)$ is defined as a set of all restrictions u of the functions in $H_p^s(\mathbb{R}^n)$ to Ω. That is, a function $u \in L_p(\Omega)$ is in $H_p^s(\Omega)$ if and only if there exists a function $U \in H_p^s(\mathbb{R}^n)$ such that $U_{|\Omega} = u$ almost everywhere in Ω. For $u \in H_p^s(\Omega)$, its H_p^s norm is defined by

$$\|u\|_{H_p^s(\Omega)} = \inf_{\substack{U_{|\Omega}=u \\ U \in H_p^s(\mathbb{R}^n)}} \|U\|_{H_p^s(\mathbb{R}^n)}.$$

By this norm, $H_p^s(\Omega)$ becomes a Banach space. When $s = 0, 1, 2, \ldots$, $H_p^s(\Omega)$ is characterized as

$$H_p^s(\Omega) = \{u \in L_p(\Omega);\ D^\alpha u \in L_p(\Omega) \quad \text{for all} \quad |\alpha| \leq s\}$$

with the norm

$$\|u\|_{H_p^s} = \sum_{|\alpha| \leq s} \|D^\alpha u\|_{L_p},$$

where $D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$ denotes the differentiation in the sense of distribution.

Let $\mathcal{D}(\Omega)$ denote the space of all $\mathcal{C}^\infty$ functions with compact support in Ω. By $\overset{\circ}{H}{}^s_p(\Omega)$, where $0 \le s < \infty$, we denote the closure of $\mathcal{D}(\Omega)$ in the space $H^s_p(\Omega)$. It is then known that

$$\overset{\circ}{H}{}^s_p(\Omega) = H^s_p(\Omega) \qquad \text{if} \quad 0 \le s \le \tfrac{1}{p}. \tag{3.2}$$

For $\frac{1}{p} < s < \infty$, the trace operator $\gamma : u \mapsto u_{|\partial\Omega}$ is defined. It is known that

$$\gamma : H^s(\Omega) \to L_p(\partial\Omega) \quad \text{is a bounded operator if} \quad \tfrac{1}{p} < s < \infty. \tag{3.3}$$

Moreover, when Ω is a bounded domain with $\mathcal{C}^2$ boundary, it holds that

$$\gamma : H^s_p(\Omega) \to H_p^{s-\frac{1}{p}}(\partial\Omega) \quad \text{is a bounded operator if} \quad \tfrac{1}{p} < s \le 2. \tag{3.4}$$

The space $\overset{\circ}{H}{}^s_p(\Omega)$ can then be characterized by

$$\overset{\circ}{H}{}^s_p(\Omega) = \{u \in H^s_p(\Omega);\ \gamma u = 0 \quad \text{on} \quad \partial\Omega\} \qquad \text{for} \quad \tfrac{1}{p} < s \le 1. \tag{3.5}$$

Let p' be the adjoint exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. For $0 < s < \infty$, $H_p^{-s}(\Omega)$ can now be defined by

$$H_p^{-s}(\Omega) = \overset{\circ}{H}{}^s_{p'}(\Omega)^*, \qquad 0 < s < \infty.$$

From $\mathcal{D}(\Omega) \subset \overset{\circ}{H}{}^s_{p'}(\Omega)$ (densely), it follows that $H_p^{-s}(\Omega) = \overset{\circ}{H}{}^s_{p'}(\Omega)^* \subset \mathcal{D}(\Omega)^*$; this means that $H_p^{-s}(\Omega)$ consists of suitable distributions in Ω. Similarly it holds that

$$\langle u, f\rangle_{L_p \times L_{p'}} = \langle u, f\rangle_{\overset{\circ}{H}{}^s_p \times H^{-s}_{p'}} \qquad \text{for} \quad u \in \overset{\circ}{H}{}^s_p(\Omega),\ f \in H_{p'}^{-s}(\Omega). \tag{3.6}$$

In this way we have defined a family of spaces $H^s_p(\Omega)$ parameterized by $-\infty < s < \infty$. Of course, when $\Omega = \mathbb{R}^n$, such a definition is consistent with (3.1). For $p = 2$, $H^s_2(\Omega) = H^s(\Omega)$ is called the Sobolev space. In the meantime, for $1 < p < \infty$, $p \ne 2$, $H^s_p(\Omega)$ is called the Lebesgue space.

We have $H^{s_1}_p(\Omega) \subset H^{s_0}_p(\Omega)$ with dense and continuous embedding for any $-\infty < s_0 < s_1 < \infty$. In addition, for $0 \le s_0 < s_1 < \infty$ and $0 \le \theta \le 1$,

$$[H^{s_0}_p(\Omega), H^{s_1}_p(\Omega)]_\theta = H_p^{(1-\theta)s_0 + \theta s_1}(\Omega). \tag{3.7}$$

This property yields that, for any function $a \in \mathcal{C}^1(\overline{\Omega})$, the multiplicative operator $u \mapsto au$ is a bounded operator from $H^s_p(\Omega)$ into itself for $-1 \le s \le 1$ with the estimate

$$\|au\|_{H^s_p} \le C\|a\|_{\mathcal{C}^1}\|u\|_{H^s_p}, \qquad u \in H^s_p(\Omega);\ -1 \le s \le 1. \tag{3.8}$$

Finally the following fundamental result is valid:

$$D_{x_j} : H^s_p(\Omega) \to H^{s-1}_p(\Omega) \quad \text{is a bounded operator} \tag{3.9}$$

for any $-\infty < s < \infty$, $s \ne \frac{1}{p}$.

4. Elliptic operators in L_2 spaces

Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Consider a sesquilinear form

$$a(u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) D_i u D_j \overline{v}\, dx$$

defined for $(u,v) \in \overset{\circ}{H}{}^1(\Omega) \times \overset{\circ}{H}{}^1(\Omega)$. Assume that $a_{ij}(x)$, $1 \le i, j \le n$, are real-valued functions satisfying

$$a_{ij} \in \mathcal{C}^1(\overline{\Omega}), \qquad 1 \le i,\, j \le 1, \tag{4.1}$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \delta|\xi|^2, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \tag{4.2}$$

with some constant $\delta > 0$. It is then clear that $a(u,v)$ is continuous, i.e., $|a(u,v)| \le C\|u\|_{H^1}\|v\|_{H^1}$ for $(u,v) \in \overset{\circ}{H}{}^1(\Omega) \times \overset{\circ}{H}{}^1(\Omega)$, and is coercive, i.e., $\operatorname{Re} a(u,u) \ge c\|u\|_{H^1}$ for $u \in \overset{\circ}{H}{}^1(\Omega)$ with some constant $c > 0$.

Then the associated linear operator with $a(u,v)$ is determined by the relation: $a(u,v) = \langle \mathcal{A}u, v\rangle_{H^{-1}\times \overset{\circ}{H}{}^1}$ for all $v \in \overset{\circ}{H}{}^1(\Omega)$, $\mathcal{A}$ being an isomorphism from $\overset{\circ}{H}{}^1(\Omega)$ onto $H^{-1}(\Omega)$. Furthermore we consider the part of $\mathcal{A}$ in $L_2(\Omega)$ which is determined by $\mathcal{D}(A) = \{u \in \overset{\circ}{H}{}^1(\Omega); \mathcal{A}u \in L_2(\Omega)\}$ and $Au = \mathcal{A}u$ for $u \in \mathcal{D}(A)$. This means that $u \in \mathcal{D}(A)$ if and only if $a(u,v)$ is continuous with respect to vector variable $v \in \overset{\circ}{H}{}^1(\Omega)$ in the topology of $L_2(\Omega)$ (which is of course weaker than $\overset{\circ}{H}{}^1(\Omega)$). It is then shown by the theory of sesquilinear forms, cf. [9, Chapter VII] or [41, Chapitre 2], that A is a sectorial operator of $L_2(\Omega)$ with angle $\omega_A < \frac{\pi}{2}$; moreover, A is a maximal accretive operator of $L_2(\Omega)$. The operator A is considered as a realization of the elliptic operator $-\sum_{i,j=1}^n D_j[a_{ij}(x)D_i u]$ in $L_2(\Omega)$ under the Dirichlet boundary conditions $\gamma u = 0$ on $\partial\Omega$.

Case of $\mathcal{C}^2$ domains. Let Ω have a boundary of $\mathcal{C}^2$ class. In this case, it is known by Grisvard [28, Chapter 2] that

$$\mathcal{D}(A) = H^2(\Omega) \cap \overset{\circ}{H}{}^1(\Omega) = \{u \in H^2(\Omega); \gamma u = 0 \text{ on } \partial\Omega\}. \tag{4.3}$$

In view of this fact and (3.2), for $\frac{1}{2} < s \le 2$, we will define

$$H_D^s(\Omega) = \{u \in H^s(\Omega); \gamma u = 0 \text{ on } \partial\Omega\}. \tag{4.4}$$

Our first characterization is then given by the following theorem.

Theorem 4.1. *Let* (4.1) *and* (4.2) *be satisfied. Then,*

$$\mathcal{D}(A^\theta) = [L_2(\Omega), H_D^2(\Omega)]_\theta = \begin{cases} H^{2\theta}(\Omega) & \text{if } 0 \le \theta < \frac{1}{4}, \\ H_D^{2\theta}(\Omega) & \text{if } \frac{1}{4} < \theta \le 1 \end{cases}$$

with norm equivalence.

Proof. The proof will be divided into four Steps.

I) As mentioned, it is already known that $\mathcal{D}(A) = H^2_D(\Omega) \subset H^2(\Omega)$ with the estimate $\|u\|_{H^2} \le C\|Au\|_{L_2}$, $u \in \mathcal{D}(A)$. We know also by Proposition 2.4 and Theorem 2.1 that A enjoys Property (4) of Theorem 2.1, i.e., $\mathcal{D}(A^\theta) = [L_2(\Omega), H^2_D(\Omega)]_\theta$. Then, (3.7) ($s_0 = 0$ and $s_1 = 2$) provides that $\mathcal{D}(A^\theta) \subset H^{2\theta}(\Omega)$ for all $0 \le \theta \le 1$ with the estimate

$$\|u\|_{H^{2\theta}} \le C_\theta \|A^\theta u\|_{L_2}, \qquad u \in \mathcal{D}(A^\theta). \tag{4.5}$$

Now let $\frac{1}{4} < \theta \le 1$ and $u \in \mathcal{D}(A^\theta)$. By the density of $\mathcal{D}(A)$ in $\mathcal{D}(A^\theta)$, there exists a sequence $u_k \in \mathcal{D}(A)$ such that $u_k \to u$ in $\mathcal{D}(A^\theta)$ as $k \to \infty$. Since $2\theta > \frac{1}{2}$, we observe by (3.3) that

$$\|\gamma u\|_{L^2(\partial\Omega)} = \|\gamma(u-u_k)\|_{L^2(\partial\Omega)} \le C\|u-u_k\|_{H^{2\theta}(\Omega)} \le C_\theta\|A^\theta(u-u_k)\|_{L^2(\Omega)} \to 0.$$

Therefore, $\gamma u = 0$ on $\partial\Omega$. Thus, we have verified that $\mathcal{D}(A^\theta) \subset H^{2\theta}_D(\Omega)$ for $\frac{1}{4} < \theta \le 1$.

II) Conversely, let us prove that $H^{2\theta}(\Omega) \subset \mathcal{D}(A^\theta)$ for $0 < \theta < \frac{1}{4}$. Let $u \in H^{2\theta}(\Omega)$ and $v \in \mathcal{D}(A^*)$. By the same reason as A, we have $\mathcal{D}(A^*) = H^2_D(\Omega)$. Using the formula

$$A^{*(\theta-1)} = \frac{1}{2\pi i}\int_\Gamma \lambda^{\theta-1}(\lambda - A^*)^{-1}d\lambda = \frac{1}{2\pi\theta i}\int_\Gamma \lambda^\theta(\lambda - A^*)^{-2}d\lambda,$$

we can write

$$\begin{aligned}
(u, A^{*\theta}v) = (u, A^{*(\theta-1)}A^*v) &= \frac{1}{2\pi\theta i}\int_\Gamma \lambda^\theta(u, A^*(\lambda-A^*)^{-2}v)d\lambda \\
&= \frac{-1}{2\pi\theta i}\int_\Gamma \lambda^\theta \sum_{i,j=1}^n (u, D_i[a_{ij}(x)D_j(\lambda-A^*)^{-2}v])d\lambda \qquad (4.6)\\
&= \frac{-1}{2\pi\theta i}\int_\Gamma \lambda^\theta \sum_{i,j=1}^n (u, D_i[a_{ij}(x)D_jA^{*(\theta-1)}A^{*(1-\theta)}(\lambda-A^*)^{-2}v])d\lambda.
\end{aligned}$$

Here we notice the following lemma.

Lemma 4.2. *It holds that*

$$|(u, D_i[a_{ij}(x)D_jA^{*(\theta-1)}f])| \le C_\theta\|u\|_{H^{2\theta}}\|f\|_{L_2} \quad \textit{for all } f \in \mathcal{D}(A^*).$$

Proof of the lemma. Since $0 < 2\theta < \frac{1}{2}$, we see that $u \in H^{2\theta}(\Omega) = \overset{\circ}{H}{}^{2\theta}(\Omega)$ (due to (3.2)). In the meantime, $A^{*(\theta-1)}$ is a bounded operator from $L_2(\Omega)$ into $H^{2(1-\theta)}(\Omega)$, since $\mathcal{D}(A^{*(1-\theta)}) \subset H^{2(1-\theta)}(\Omega)$ (due to (4.5) applied for A^*). The differentiation D_j is a bounded operator from $H^{2(1-\theta)}(\Omega)$ to $H^{1-2\theta}(\Omega)$ (due to (3.9)), the multiplication of $a_{ij}(x)$ is a bounded operator on $H^{1-2\theta}(\Omega)$ (due to (3.8)), and D_i is a bounded operator from $H^{1-2\theta}(\Omega)$ to $H^{-2\theta}(\Omega)$ because of $1-2\theta \ne \frac{1}{2}$ (due to (3.9) again). So, $f \mapsto D_i[a_{ij}(x)D_jA^{*(\theta-1)}f]$ is a bounded operator from $L_2(\Omega)$

into $H^{-2\theta}(\Omega)$. Therefore, in view of (3.6), we obtain that

$$(u, D_i[a_{ij}(x)D_jA^{*(\theta-1)}f]) = \langle u, D_i[a_{ij}(x)D_jA^{*(\theta-1)}f]\rangle_{\overset{\circ}{H}^{2\theta}\times H^{-2\theta}},$$

as well as the desired estimate. □

Lemma 4.2 provides existence of a function $\widetilde{u}_{ij} \in L_2(\Omega)$ such that

$$(u, D_i[a_{ij}(x)D_jA^{*(\theta-1)}f]) = (\widetilde{u}_{ij}, f) \qquad \text{for all } f \in \mathcal{D}(A^*)$$

with the estimate

$$\|\widetilde{u}_{ij}\|_{L_2} \le C_\theta\|u\|_{H^{2\theta}}, \qquad 1 \le i,\, j \le n.$$

Furthermore, using $\widetilde{u}_{ij}$, we can write

$$\begin{aligned}(u, A^{*\theta}v) &= \frac{-1}{2\pi\theta i}\int_\Gamma \lambda^\theta(\textstyle\sum_{i,j=1}^n \widetilde{u}_{ij}, A^{*(1-\theta)}(\lambda - A^*)^{-2}v)d\lambda\\ &= \frac{-1}{2\pi\theta i}\int_\Gamma \lambda^\theta(A^{1-\theta}(\overline{\lambda} - A)^{-2}[\textstyle\sum_{i,j=1}^n \widetilde{u}_{ij}], v)d\lambda.\end{aligned}$$

Hence, we can use (2.5) to conclude that

$$|(u, A^{*\theta}v)| \le C_\theta\|u\|_{H^{2\theta}}\|v\|_{L_2}. \tag{4.7}$$

Since $v \in \mathcal{D}(A^*)$ is arbitrary and $\mathcal{D}(A^*)$ is dense in $L_2(\Omega)$, it follows that $u \in \mathcal{D}([A^{*\theta}]^*) = \mathcal{D}(A^\theta)$ with $\|A^\theta u\|_{L_2} \le C_\theta\|u\|_{H^{2\theta}}$.

III) Let us next prove that $H^{2\theta}_D(\Omega) \subset \mathcal{D}(A^\theta)$ for $\frac{1}{4} < \theta \le \frac{1}{2}$. Let $u \in H^{2\theta}_D(\Omega)$ and let $v \in \mathcal{D}(A^*)$. In the present case, we use the expression

$$\begin{aligned}(u, A^{*\theta}v) = (u, A^{*(\theta-1)}A^*v) &= \frac{1}{2\pi\theta i}\int_\Gamma \lambda^\theta(u, A^*(\lambda - A^*)^{-2}v)d\lambda\\ &= \frac{1}{2\pi\theta i}\int_\Gamma \lambda^\theta\sum_{i,j=1}^n(a_{ij}(x)D_iu, D_jA^{*(\theta-1)}A^{*(1-\theta)}(\lambda - A^*)^{-2}v)d\lambda.\end{aligned}$$

We notice the following lemma.

Lemma 4.3. *It holds that*

$$|(a_{ij}(x)D_iu, D_jA^{*(\theta-1)}f)| \le C_\theta\|u\|_{H^{2\theta}}\|f\|_{L_2} \quad \textit{for all } f \in \mathcal{D}(A^*).$$

Proof of the lemma. Since $2\theta \ne \frac{1}{2}$, we see that $D_iu \in H^{2\theta-1}(\Omega)$ (due to (3.9)). In addition, the multiplication of $a_{ij}(x)$ is a bounded operator on $H^{2\theta-1}(\Omega)$ (due to (3.8)). So, $a_{ij}(x)D_iu \in H^{2\theta-1}(\Omega)$. In the meantime, $A^{*(\theta-1)}$ is a bounded operator from $L_2(\Omega)$ into $H^{2(1-\theta)}(\Omega)$ as before and D_j is a bounded operator from $H^{2(1-\theta)}(\Omega)$ into $H^{1-2\theta}(\Omega)$ (due to (3.9)). So, $f \mapsto D_jA^{*(\theta-1)}f$ is a bounded operator from $L_2(\Omega)$ into $H^{1-2\theta}(\Omega)$. Furthermore, $H^{1-2\theta}(\Omega) = \overset{\circ}{H}^{1-2\theta}(\Omega)$ because of $0 \le 1 - 2\theta < \frac{1}{2}$. Hence, in view of (3.6), we obtain that

$$(a_{ij}(x)D_iu, D_jA^{*(\theta-1)}f) = \langle a_{ij}(x)D_iu, D_jA^{*(\theta-1)}f\rangle_{H^{2\theta-1}\times\overset{\circ}{H}^{1-2\theta}}$$

and as well the estimate to be shown. □

As before, Lemma 4.3 provides existence of a function $\widetilde{u}_{ij} \in L_2(\Omega)$ such that

$$(a_{ij}(x)D_i u, D_j A^{*(\theta-1)} f) = (\widetilde{u}_{ij}, f) \qquad \text{for all } f \in \mathcal{D}(A^*)$$

with the estimate

$$\|\widetilde{u}_{ij}\|_{L_2} \le C_\theta \|u\|_{H^{2\theta}}, \qquad 1 \le i,\, j \le n.$$

Then, by the same arguments appealing to (2.5), we can conclude that (4.7) is valid in this case too, and consequently $u \in \mathcal{D}(A^\theta)$.

IV) Finally, let us prove that $H_D^{2\theta}(\Omega) \subset \mathcal{D}(A^\theta)$ for $\frac{1}{2} \le \theta \le 1$. We already know that $\mathcal{D}(A^{\frac{1}{2}}) = H_D^1(\Omega)$ and $\mathcal{D}(A) = H_D^2(\Omega)$. By Proposition 2.4 and Theorem 2.1, $\mathcal{D}(A^\theta) = [\mathcal{D}(A^{\frac{1}{2}}), \mathcal{D}(A)]_{2\theta-1}$. Meanwhile, we can show by the following lemma that $[H_D^1(\Omega), H_D^2(\Omega)]_{2\theta-1} = H_D^{2\theta}(\Omega)$. Hence, $H_D^{2\theta}(\Omega) = \mathcal{D}(A^\theta)$.

Lemma 4.4. *For any* $0 \le \eta \le 1$*, it holds that* $[H_D^1(\Omega), H_D^2(\Omega)]_\eta = H_D^{1+\eta}(\Omega)$ *with norm equivalence.*

Proof of the lemma. Before proving the lemma, we will construct a right inverse of γ which is continuous from $H^{i-\frac{1}{2}}(\partial\Omega)$ into $H^i(\Omega)$ for $i = 1, 2$. Consider the elliptic boundary value problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \gamma u = g & \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

For any $g \in H^{\frac{1}{2}}(\partial\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ to (4.8). Indeed, let r be a continuous right inverse of γ; due to (3.4), r is a bounded linear operator from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^1(\Omega)$ such that $\gamma(rg) = g$ for all $g \in H^{\frac{1}{2}}(\partial\Omega)$. Since $\Delta(rg) \in H^{-1}(\Omega)$, there exists a unique function $u_D \in \overset{\circ}{H}{}^1(\Omega)$ such that $-\Delta u_D = \Delta(rg)$. Then, since $\overset{\circ}{H}{}^1(\Omega) = H_D^1(\Omega)$, $u = u_D + rg$ gives the desired solution. Define an operator R by $Rg = u = u_D + rg$ for $g \in H^{\frac{1}{2}}(\Omega)$. Then R is another continuous right inverse of γ from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^1(\Omega)$. Moreover, by the regularity of solution of (4.8), we can equally verify that R is continuous from $H^{\frac{3}{2}}(\partial\Omega)$ into $H^2(\Omega)$.

Let us now consider the proof of lemma. As $H_D^i(\Omega) \subset H^i(\Omega)$ for $i = 1, 2$, we have $[H_D^1(\Omega), H_D^2(\Omega)]_\eta \subset [H^1(\Omega), H^2(\Omega)]_\eta = H^{1+\eta}(\Omega)$. Furthermore, as it is clear that $[H_D^1(\Omega), H_D^2(\Omega)]_\eta \subset H_D^1(\Omega)$, we verify by the definition (4.4) that $[H_D^1(\Omega), H_D^2(\Omega)]_\eta \subset H_D^{1+\eta}(\Omega)$.

Conversely, let $u \in H_D^{1+\eta}(\Omega)$. By the definition of complex interpolation, there exists an analytic function $F(z)$ defined in the strip domain $\{z; 0 < \operatorname{Re} z < 1\}$ with values in $H^1(\Omega)$ which is bounded and continuous for $0 \le \operatorname{Re} z \le 1$ with values in $H^1(\Omega)$ and is bounded and continuous for $z = 1 + iy\,(y \in \mathbb{R})$ with values in $H^2(\Omega)$ and which has its value $F(\eta) = u$ at $z = \eta$. Consider the analytic function $F_D(z) = F(z) - R[\gamma F(z)]$. Clearly, $F_D(z)$ has its values in $H_D^1(\Omega)$ and $F_D(1 + iy)$ has its values in $H_D^2(\Omega)$; in addition, $F_D(\eta) = u$. This means that $u \in [H_D^1(\Omega), H_D^2(\Omega)]_\eta$. Hence, $H_D^{1+\eta}(\Omega) \subset [H_D^1(\Omega), H_D^2(\Omega)]_\eta$. □

We have thus accomplished the proof of theorem. □

Case of convex domains. Let Ω be a bounded convex domain in $\mathbb{R}^n$. In this case we add the assumption that $a_{ij}(x)$ are symmetric in the sense that

$$a_{ij}(x) = a_{ji}(x), \qquad 1 \le i,\, j \le n. \tag{4.9}$$

Under this condition, A is a self-adjoint operator of $L_2(\Omega)$. It is also known by Grisvard [28, Chapter 3] that

$$\mathcal{D}(A) = H^2_D(\Omega). \tag{4.10}$$

We then obtain a similar characterization of domains of fractional powers as Theorem 4.1.

Theorem 4.5. *Let* (4.1)*,* (4.2) *and* (4.9) *be satisfied. Then,*

$$\mathcal{D}(A^\theta) = [L_2(\Omega), H^2_D(\Omega)]_\theta = \begin{cases} H^{2\theta}(\Omega) & \text{if } 0 \le \theta < \frac{1}{4}, \\ H^{2\theta}_D(\Omega) & \text{if } \frac{1}{4} < \theta \le 1, \theta \ne \frac{3}{4} \end{cases}$$

with norm equivalence.

Proof. I) Since (4.10) is the case as (4.3), the same arguments as in Step 1 of proof of Theorem 4.1 is still available in the present case to conclude that $\mathcal{D}(A^\theta) \subset H^{2\theta}(\Omega)$ for $0 \le \theta < \frac{1}{4}$ and $\mathcal{D}(A^\theta) \subset H^{2\theta}_D(\Omega)$ for $\frac{1}{4} < \theta \le 1$.

II) Similarly, we can repeat the same arguments as in Step 2 of proof of Theorem 4.1 to conclude that $H^{2\theta}(\Omega) \subset \mathcal{D}(A^\theta)$ for $0 \le \theta < \frac{1}{4}$.

III) In Step 3 of proof of Theorem 4.1, we proved that $H^{2\theta}_D(\Omega) \subset \mathcal{D}(A^\theta)$ for $\frac{1}{4} < \theta \le \frac{1}{2}$. We easily observe that the method of this proof is available even for $\frac{1}{2} \le \theta < \frac{3}{4}$, too, for it holds that $\overset{\circ}{H}{}^{2\theta-1}(\Omega) = H^{2\theta-1}(\Omega)$. In this way, we can verify that the inclusion $H^{2\theta}_D(\Omega) \subset \mathcal{D}(A^\theta)$ is true for any exponent $\frac{1}{4} < \theta < \frac{3}{4}$.

IV) It now remains to prove that $H^{2\theta}_D(\Omega) \subset \mathcal{D}(A^\theta)$ for $\frac{3}{4} < \theta < 1$. For $u \in H^{2\theta}_D(\Omega)$ and $v \in \mathcal{D}(A)$, we have the expression

$$\begin{aligned}(u, A^\theta v) = (u, A^{\theta-1}Av) &= \frac{1}{2\pi\theta i}\int_\Gamma \lambda^\theta (u, A(\lambda - A)^{-2}v)d\lambda \\ &= \frac{-1}{2\pi\theta i}\int_\Gamma \lambda^\theta \sum_{i,j=1}^n (D_j[a_{ij}(x)D_i u], A^{\theta-1}A^{1-\theta}(\lambda - A)^{-2}v)d\lambda.\end{aligned}$$

Here the following lemma is observed.

Lemma 4.6. $|(D_j[a_{ij}(x)D_i u], A^{\theta-1}f)| \le C_\theta \|u\|_{H^{2\theta}} \|f\|_{L_2}$ *for all* $f \in \mathcal{D}(A)$.

Proof of the lemma. We notice that D_i is a bounded operator from $H^{2\theta}(\Omega)$ into $H^{2\theta-1}(\Omega)$ (due to (3.9)), the multiplication of $a_{ij}(x)$ is a bounded operator on $H^{2\theta-1}(\Omega)$ (due to (3.8)), and D_j is a bounded operator from $H^{2\theta-1}(\Omega)$ into $H^{2(\theta-1)}(\Omega)$ because of $2\theta - 1 \ne \frac{1}{2}$ (due to again (3.9)). So, $D_j[a_{ij}(x)D_i u] \in H^{2(\theta-1)}(\Omega)$. In the meantime, as shown in Step 1, $A^{\theta-1}$ is a bounded operator from $L_2(\Omega)$ into $H^{2(1-\theta)}(\Omega)$. Furthermore, $H^{2(1-\theta)}(\Omega) = \overset{\circ}{H}{}^{2(1-\theta)}(\Omega)$ because of $0 \le 2(1-\theta) < \frac{1}{2}$. Hence, in view of (3.6), we have

$$(D_j[a_{ij}(x)D_i u], A^{\theta-1}f) = \langle D_j[a_{ij}(x)D_i u], A^{\theta-1}f\rangle_{H^{2(\theta-1)} \times \overset{\circ}{H}{}^{2(1-\theta)}}.$$

Thus, the desired estimate is obtained. □

This lemma provides existence of a function $\widetilde{u}_{ij} \in L_2(\Omega)$ such that

$$(D_j[a_{ij}(x)D_i u], A^{\theta-1}f) = (\widetilde{u}_{ij}, f) \qquad \text{for all } f \in \mathcal{D}(A)$$

with the estimate

$$\|\widetilde{u}_{ij}\|_{L_2} \le C_\theta \|u\|_{H^{2\theta}}.$$

Then the same arguments as in the proof of Theorem 4.1 provides the desired inclusion $H_D^{2\theta}(\Omega) \subset \mathcal{D}(A^\theta)$ for $\frac{3}{4} < \theta < 1$.

We have thus accomplished the proof of theorem. □

5. Elliptic operators in L_p spaces

In this section, we are concerned with a realization of $-\sum_{i,j=1}^n D_j[a_{ij}(x)D_i u]$ in the L_p space under the Dirichlet boundary conditions $\gamma u = 0$ on $\partial\Omega$, where $1 < p < \infty$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary. The functions $a_{ij}(x)$, $1 \le i, j \le n$, are all real-valued and are assumed to satisfy the conditions (4.1) and (4.2). Let A_2 be the realization of $-\sum_{i,j=1}^n D_j[a_{ij}(x)D_i u]$ in $L_2(\Omega)$ under the Dirichlet boundary conditions on $\partial\Omega$ which have been treated in the preceding section. When $2 < p < \infty$, the realization A_p of the elliptic operator in $L_p(\Omega)$ is defined by $\mathcal{G}(A_p) = \mathcal{G}(A_2) \cap [L_p(\Omega) \times L_p(\Omega)]$, where $\mathcal{G}(A_p)$ denotes the graph of A_p. Meanwhile, when $1 < p < 2$, A_p is defined by $\mathcal{G}(A_p) = \overline{\mathcal{G}(A_2)}$ in the product space $L_p(\Omega) \times L_p(\Omega)$. According to Grisvard [28, Chapter 2], it is known for any $1 < p < \infty$ that A_p is a sectorial operator of $L_p(\Omega)$ and its domain is given by

$$\mathcal{D}(A_p) = \{u \in H_p^2(\Omega);\ \gamma u = 0 \text{ on } \partial\Omega\}. \tag{5.1}$$

Furthermore, it holds that

$$\widetilde{\delta}_p \|u\|_{H_p^2} \le \|A_p u\|_{L_p}, \qquad u \in \mathcal{D}(A_p)$$

with some constant $\delta_p > 0$. In view of (3.3) , for $\frac{1}{p} < s \le 2$, we will put

$$H_{p,D}^s = \{u \in H_p^s(\Omega);\ \gamma u = 0 \text{ on } \partial\Omega\}.$$

Let p' be the adjoint exponent such that $\frac{1}{p} + \frac{1}{p'} = 1$, and let $\{L_p(\Omega), L_{p'}(\Omega)\}$ be the adjoint pair. The adjoint operator of A_p with respect to $\{L_p(\Omega), L_{p'}(\Omega)\}$ is then defined; let us denote it by $B_{p'} = A_p^*$; then, $B_{p'}$ is seen to be a realization of the differential operator $-\sum_{i,j=1}^n D_i[a_{ij}(x)D_j v]$ in $L_{p'}(\Omega)$ under the same Dirichlet boundary conditions on $\partial\Omega$. Consequently, $B_{p'}$ is a sectorial operator of $L_{p'}(\Omega)$ with domain

$$\mathcal{D}(B_{p'}) = \{v \in H_{p'}^2(\Omega);\ \gamma v = 0 \text{ on } \partial\Omega\}.$$

The estimate

$$\delta^*_{p'}\|v\|_{H^2_{p'}} \le \|B_{p'}v\|_{L_{p'}}, \qquad v \in \mathcal{D}(B_{p'})$$

is still true with some constant $\delta^*_{p'} > 0$.

Concerning the domains of fractional powers A^θ, $0 \le \theta \le 1$, we can show the following two kinds of results.

Theorem 5.1. *Let* (4.1) *and* (4.2) *be satisfied. In addition, we assume that* A_p *has bounded imaginary powers in* $L_p(\Omega)$, *i.e.,* $\|A_p^{iy}\|_{\mathcal{L}(L_p)} \le N_p e^{\beta_p|y|}$, $-\infty < y < \infty$. *Then,*

$$\mathcal{D}(A_p^\theta) \subset \begin{cases} H_p^{2\theta}(\Omega) & \text{if} \quad 0 \le \theta < \frac{1}{2p}, \\ H_{p,D}^{2\theta}(\Omega) & \text{if} \quad \frac{1}{2p} < \theta \le 1 \end{cases}$$

with continuous embedding.

Proof. It is possible to carry out the proof in a quite analogous way as in Step 1 of proof of Theorem 4.1. In fact, by Theorem 2.2 and (5.1), we have

$$\mathcal{D}(A_p^\theta) = [L_p(\Omega), \mathcal{D}(A_p)]_\theta \subset [L_p(\Omega), H_p^2(\Omega)]_\theta.$$

Meanwhile, by (3.7), $[L_p(\Omega), H_p^2(\Omega)]_\theta = H_p^{2\theta}(\Omega)$. Therefore, $\mathcal{D}(A_p^\theta) \subset H^{2\theta}(\Omega)$ for every $0 \le \theta \le 1$ with continuous embedding

$$\|u\|_{H_p^{2\theta}} \le C_p\|A_p^\theta u\|_{L_p}, \qquad u \in \mathcal{D}(A_p^\theta). \tag{5.2}$$

Let $\frac{1}{2p} < \theta \le 1$. As $\mathcal{D}(A_p)$ is dense in $\mathcal{D}(A_p^\theta)$, for any $u \in \mathcal{D}(A_p^\theta)$, there is a sequence $u_k \in \mathcal{D}(A_p)$ such that $u_k \to u$ in $\mathcal{D}(A_p^\theta)$. This together with (3.3) and (5.2) then implies that u satisfies the boundary conditions $\gamma u = 0$. Hence, $\mathcal{D}(A_p^\theta) \subset H_{p,D}^{2\theta}(\Omega)$. □

Theorem 5.2. *Let* (4.1) *and* (4.2) *be satisfied. Assume that* A_p *has a bounded* H_∞ *functional calculus in* $L_p(\Omega)$, *namely,* $\|f(A_p)\|_{\mathcal{L}(L_p)} \le C_\omega\|f\|_\infty$, $f \in H_\infty(\Sigma_\omega)$, *for any angle* ω *such that* $\omega_{A_p} < \omega \le \pi$, *where* ω_{A_p} *denotes the angle of* A_p. *Then,*

$$\mathcal{D}(A_p^\theta) = \begin{cases} H_p^{2\theta}(\Omega) & \text{if} \quad 0 \le \theta < \frac{1}{2p}, \\ H_{p,D}^{2\theta}(\Omega) & \text{if} \quad \frac{1}{2p} < \theta \le 1,\ \theta \ne \frac{p+1}{2p} \end{cases}$$

with norm equivalence.

Proof. According to Theorem 2.2, the bounded H_∞ functional calculus implies the boundedness of the imaginary powers. Therefore, Theorem 5.1 provides that $\mathcal{D}(A_p^\theta) \subset H_p^{2\theta}(\Omega)$ for $0 \le \theta < \frac{1}{2p}$ and $\mathcal{D}(A_p^\theta) \subset H_{p,D}^{2\theta}(\Omega)$ for $\frac{1}{2p} < \theta \le 1$ with continuous embedding. So it suffices to prove the converse relations of inclusion for the domains $\mathcal{D}(A_p^\theta)$ and the spaces $H_p^{2\theta}(\Omega)$. But this is carried out in a quite analogous way as in the proof of Theorems 4.1 and 4.5.

I) Let us first verify that $H_p^{2\theta}(\Omega) \subset \mathcal{D}(A_p^\theta)$ for $0 \le \theta < \frac{1}{2p}$. Since $B_{p'}^{iy} = (A_p^{-iy})^*$ with respect to the adjoint pair $\{L_p(\Omega), L_{p'}(\Omega)\}$, Theorem 5.1 is available also for the adjoint operator $B_{p'}$. As a result, for every $0 \le \theta \le 1$,

$$\|v\|_{H^{2\theta}_{p'}} \le C_{p'} \|B^{\theta}_{p'} v\|_{L_{p'}}, \qquad v \in \mathcal{D}(B^{\theta}_{p'}). \tag{5.3}$$

Let $u \in H^{2\theta}_p(\Omega)$ and $v \in \mathcal{D}(B_{p'})$. As for (4.6), we have the expression

$$\begin{aligned}\langle u, B^{\theta}_{p'} v\rangle_{L_p \times L_{p'}} &= \frac{-1}{2\pi\theta i} \int_{\Gamma} \lambda^{\theta} \sum_{i,j=1}^{n} \langle u, D_i[a_{ij}(x) D_j (\lambda - B_{p'})^{-2} v]\rangle_{L_p \times L_{p'}} d\lambda \\ &= \frac{-1}{2\pi\theta i} \int_{\Gamma} \lambda^{\theta} \sum_{i,j=1}^{n} \langle u, D_i[a_{ij}(x) D_j B^{\theta-1}_{p'} B^{1-\theta}_{p'} (\lambda - B_{p'})^{-2} v]\rangle_{L_p \times L_{p'}} d\lambda.\end{aligned}$$

Since $0 < 2\theta < \frac{1}{p}$, we see that $u \in H^{2\theta}_p(\Omega) = \overset{\circ}{H}{}^{2\theta}_p(\Omega)$ due to (3.2). Then, by the same arguments as in Step 2 of proof of Theorem 4.2, we can obtain the estimate

$$\begin{aligned}|\langle u, D_i[a_{ij}(x) D_j B^{\theta-1}_{p'} f]\rangle_{L_p \times L_{p'}}| &= |\langle u, D_i[a_{ij}(x) D_j B^{\theta-1}_{p'} f]\rangle_{\overset{\circ}{H}{}^{2\theta}_p \times H^{-2\theta}_{p'}}| \\ &\le C_{\theta} \|u\|_{H^{2\theta}_p} \|f\|_{L_{p'}} \qquad \text{for all } f \in L_{p'}(\Omega).\end{aligned}$$

As a consequence, there exists a function $\widetilde{u}_{ij} \in L_p(\Omega)$ such that

$$\langle u, D_i[a_{ij}(x) D_j B^{\theta-1}_{p'} f]\rangle_{L_p \times L_{p'}} = \langle \widetilde{u}_{ij}, f\rangle_{L_p \times L_{p'}} \qquad \text{for all } f \in L_{p'}(\Omega)$$

with the estimate

$$\|\widetilde{u}_{ij}\|_{L_p} \le C_{\theta} \|u\|_{H^{2\theta}_p}.$$

Therefore,

$$\begin{aligned}\langle u, B^{\theta}_{p'} v\rangle_{L_p \times L_{p'}} &= \frac{-1}{2\pi\theta i} \int_{\Gamma} \lambda^{\theta} \langle \textstyle\sum_{i,j=1}^{n} \widetilde{u}_{ij}, B^{1-\theta}_{p'} (\lambda - B_{p'})^{-2} v\rangle_{L_p \times L_{p'}} d\lambda \\ &= \frac{-1}{2\pi\theta i} \int_{\Gamma} \lambda^{\theta} \langle A^{1-\theta}_p (\overline{\lambda} - A_p)^{-2} [\textstyle\sum_{i,j=1}^{n} \widetilde{u}_{ij}], v\rangle_{L_p \times L_{p'}} d\lambda.\end{aligned}$$

Applying (2.5), we conclude that

$$|\langle u, B^{\theta}_{p'} v\rangle_{L_p \times L_{p'}}| \le C_{\theta} \|u\|_{H^{2\theta}_p} \|v\|_{L_{p'}}.$$

We thus deduce that $u \in \mathcal{D}((B^{\theta}_{p'})^*) = \mathcal{D}(A^{\theta}_p)$ with $\|A^{\theta}_p u\|_{L_p} \le C_{\theta} \|u\|_{H^{2\theta}_p}$.

II) $H^{2\theta}_{p,D}(\Omega) \subset \mathcal{D}(A^{\theta}_p)$ for $\frac{1}{2p} < \theta < \frac{p+1}{2p}$. In fact, let $u \in H^{2\theta}_{p,D}(\Omega)$ and $v \in \mathcal{D}(B_{p'})$. In this case we use the expression

$$\begin{aligned}\langle u, B^{\theta}_{p'} v\rangle_{L_p \times L_{p'}} &= \frac{1}{2\pi\theta i} \int_{\Gamma} \lambda^{\theta} \sum_{i,j=1}^{n} \langle a_{ij}(x) D_i u, D_j (\lambda - B_{p'})^{-2} v\rangle_{L_p \times L_{p'}} d\lambda \\ &= \frac{1}{2\pi\theta i} \int_{\Gamma} \lambda^{\theta} \sum_{i,j=1}^{n} \langle a_{ij}(x) D_i u, D_j B^{\theta-1}_{p'} B^{1-\theta}_{p'} (\lambda - B_{p'})^{-2} v\rangle_{L_p \times L_{p'}} d\lambda.\end{aligned}$$

If $\frac{1}{2p} < \theta \le \frac{1}{2}$, then $0 \le 1-2\theta < \frac{1}{p'}$; therefore, $\overset{\circ}{H}{}^{1-2\theta}_{p'}(\Omega) = H^{1-2\theta}_{p'}(\Omega)$. If $\frac{1}{2} \le \theta < \frac{p+1}{2p}$, then $0 \le 2\theta - 1 < \frac{1}{p}$; therefore, $\overset{\circ}{H}{}^{2\theta-1}_{p}(\Omega) = H^{2\theta-1}_{p}(\Omega)$. Then, by the same arguments as in Step 3 of proof of Theorem 4.1, we conclude that

$$|\langle a_{ij}(x)D_iu, D_jB_{p'}^{\theta-1}f\rangle_{L_p\times L_{p'}}| \le C_\theta\|u\|_{H_p^{2\theta}}\|f\|_{L_{p'}} \qquad \text{for all } f\in L_{p'}(\Omega).$$

Hence, there exists a function $\widetilde{u}_{ij} \in L_p(\Omega)$ such that

$$\langle a_{ij}(x)D_iu, D_jB_{p'}^{\theta-1}f\rangle_{L_p\times L_{p'}} = \langle \widetilde{u}_{ij}, f\rangle_{L_p\times L_{p'}} \qquad \text{for all } f\in L_{p'}(\Omega)$$

with the estimate

$$\|\widetilde{u}_{ij}\|_{L_p} \le C_\theta\|u\|_{H_p^{2\theta}}.$$

Thus, $u\in\mathcal{D}(A_p^\theta)$ with $\|A_p^\theta u\|_{L_p} \le C_\theta\|u\|_{H_p^{2\theta}}$.

III) Let us finally prove that $H^{2\theta}_{p,D}(\Omega) \subset \mathcal{D}(A_p^\theta)$ for $\frac{p+1}{2p} < \theta \le 1$. Let $u \in H^{2\theta}_{p,D}(\Omega)$ and $v\in\mathcal{D}(B_{p'})$. Then,

$$\begin{aligned}\langle u, B_{p'}^\theta v\rangle_{L_p\times L_{p'}} &= \frac{-1}{2\pi\theta i}\int_\Gamma \lambda^\theta \sum_{i,j=1}^n \langle D_j[a_{ij}(x)D_iu], (\lambda - B_{p'})^{-2}v\rangle_{L_p\times L_{p'}}d\lambda \\ &= \frac{-1}{2\pi\theta i}\int_\Gamma \lambda^\theta \sum_{i,j=1}^n \langle D_j[a_{ij}(x)D_iu], B_{p'}^{\theta-1}B_{p'}^{1-\theta}(\lambda - B_{p'})^{-2}v\rangle_{L_p\times L_{p'}}d\lambda.\end{aligned}$$

We here observe that $D_j[a_{ij}(x)D_iu] \in H_p^{2(\theta-1)}(\Omega)$. In the meantime, $B_{p'}^{\theta-1}$ is a bounded operator from $L_{p'}(\Omega)$ into $H_{p'}^{2(1-\theta)}(\Omega)$ (due to (5.3)). Furthermore, since $0 < 2(1-\theta) < \frac{1}{p'}$, we have $H_{p'}^{2(1-\theta)}(\Omega) = \overset{\circ}{H}{}_{p'}^{2(1-\theta)}(\Omega)$. Therefore, it follows that

$$\begin{aligned}|\langle D_j[a_{ij}(x)D_iu], B_{p'}^{\theta-1}f\rangle_{L_p\times L_{p'}}| &= |\langle D_j[a_{ij}(x)D_iu], B_{p'}^{\theta-1}f\rangle_{H_p^{2(\theta-1)}\times \overset{\circ}{H}{}_{p'}^{2(1-\theta)}}| \\ &\le C_\theta\|u\|_{H_p^{2\theta}}\|f\|_{L_{p'}} \qquad \text{for all } f\in L_{p'}(\Omega).\end{aligned}$$

Hence, with a suitable function $\widetilde{u}_{ij} \in L_p(\Omega)$,

$$\langle D_j[a_{ij}(x)D_iu], B_{p'}^{\theta-1}f\rangle_{L_p\times L_{p'}} = \langle \widetilde{u}_{ij}, f\rangle_{L_p\times L_{p'}} \qquad \text{for all } f\in L_{p'}(\Omega)$$

with the estimate

$$\|\widetilde{u}_{ij}\|_{L_p} \le C_\theta\|u\|_{H_p^{2\theta}}.$$

In this way, it is deduced that $u\in\mathcal{D}(A_p^\theta)$ with $\|A_p^\theta u\|_{L_p} \le C_\theta\|u\|_{H_p^{2\theta}}$.

We have thus accomplished the proof of theorem. □

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Operators on Locally Convex Spaces

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Abstract. This paper is essentially a survey of some previous author's results, but it contains also some new concepts and theorems. In the second section we discuss the (already known) concepts of an operator algebra on a locally convex space and of a topologizable operator (see definitions below). We also introduce and study here the (new) concept of jointly topologizable operators. In the third section we discuss some questions open for all infinitely dimensional Banach spaces, such as the problem of existence of closed proper invariant and hyperinvariant subspaces for all continuous endomorphisms of a given space, and the Fell-Doran Problem in the representation theory of algebras. These problems can be solved in positive for the locally convex space (s) of all numerical sequences. The novelty here is the Theorem 3.2 stating that all continuous endomorphisms of the complex space (s), which are not scalar multiples of the identity operator, have closed proper hyperinvariant subspaces.

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1. Introduction

The aim of this paper is to show the differences between some aspects of operator theory on Banach spaces and on non-normed locally convex spaces. In Section 2 we show that many properties of the algebra $L(X)$ of all continuous endomorphisms of a locally convex space are quite different from these for a Banach space X. In Section 3 we show that some difficult problems, open for all infinite-dimensional Banach spaces, can be solved for a specific infinite-dimensional (complete) locally convex space. This paper is based upon the author's talk during IWOTA 2006.

We shall discuss the following topics:

1. Problem of topologization of the algebra of all continuous endomorphisms of a locally convex space and joint continuity of its action on the space in question.
2. Problem of existence of an infinite-dimensional (separable) locally convex space for which all continuous endomorphisms have proper closed invariant (hyperinvariant) subspaces.
3. The Fell and Doran Problem concerning representations of algebras on locally convex spaces (the problem is still open for any infinite-dimensional Banach space).

The paper is mostly a survey of the already known results, obtained in [11]–[14], it contains, however, some new results and concepts. In particular, all results concerning the spaces $L_\beta(X)$ are new, as well as Theorem 3.2 on existence of proper closed hyperinvariant subspaces for all continuous endomorphisms of a certain locally convex space.

All linear spaces and algebras considered in this paper are either complex or real (unless the field of scalars is explicitly indicated).

2. Operator algebras on locally convex spaces

One of basic facts in the operator theory of Banach spaces is that the algebra $L(X)$ of all continuous endomorphisms of such a space X has a natural topology for which it is a topological (Banach) algebra, i.e., the multiplication is jointly continuous ($\|T_1T_2\| \le \|T_1\|\,\|T_2\|$), and the action $(T,x) \mapsto Tx$ of $L(X)$ on X is jointly continuous ($\|Tx\| \le \|T\|\,\|x\|$). These two facts fail, generally speaking, for non-normed locally convex spaces.

The topology of a locally convex space X is given by means of a family $(|\cdot|_\alpha)_{\alpha\in\mathfrak{a}}$ of seminorms, which, without loss of generality, can be assumed saturated, i.e., for any finite set $|\cdot|_{\alpha_1},\ldots,|\cdot|_{\alpha_n}$ there is a seminorm $|\cdot|_{\alpha_{n+1}}$ such that

$$|x|_{\alpha_i} \le C_i|x|_{\alpha_{n+1}}, \quad 1 \le i \le n,$$

for some positive C_i and all x in X. Under this assumption for every continuous seminorm $|\cdot|$ on X there is an index α in $\mathfrak{a}$ and a positive constant C such that

$$|x| \le C|x|_\alpha, \quad x \in X. \tag{1}$$

In the case when X is metrizable, its topology can be given by means of a sequence of seminorms

$$|x|_1 \le |x|_2 \le \ldots, \quad x \in X,$$

which is obviously saturated. A locally convex space X is *complete* if every Cauchy net (x_μ) of its elements is convergent (i.e., for each $\alpha \in \mathfrak{a}$ and each $\varepsilon > 0$ there is μ_0 such that $|x_{\mu_1} - x_{\mu_2}| < \varepsilon$ whenever $\mu_1, \mu_2 \ge \mu_0$). A linear endomorphism T of X is continuous if for each $\alpha \in \mathfrak{a}$ the seminorm $x \mapsto |Tx|_\alpha$ is continuous (satisfies relation (1)). Thus T is continuous if for each $\alpha \in \mathfrak{a}$ there is a positive constant C and an index $\beta(\alpha) \in \mathfrak{a}$ such that

$$|Tx|_\alpha \leq C|x|_{\beta(\alpha)} \quad \text{for all} \quad x \in X. \tag{2}$$

An algebra A is said locally convex topologizable (shortly: LC-topologizable) if it has a locally convex topology making it a topological algebra, i.e., the multiplication there is jointly continuous. Each locally convex algebra A can be topologized by means of a family $(|\cdot|_\alpha)_{\alpha\in\mathfrak{a}}$ of seminorms such that for each α there is a $\beta(\alpha)$ with

$$|xy|_\alpha \leq |x|_{\beta(\alpha)}|y|_{\beta(\alpha)}, \quad x, y \in A. \tag{3}$$

If A has a unity e it can be assumed $|e|_\alpha = 1$ for all α in $\mathfrak{a}$. For more information about locally convex algebras the reader is referred to [6] and [9]. In contrast with the Banach space theory, the algebra $L(X)$ is rarely LC-topologizable. We have [12] the following result.

Theorem 2.1. *Let X be a sequentially complete locally convex space. Then the following are equivalent*:

(i) *The algebra $L(X)$ is LC-topologizable.*
(ii) *The space X is subnormed.*
(iii) *The space X has a maximal bounded set.*
(iv) *The algebra $L(X)$ is normable.*
(v) *The space X is bornologically normable.*

We have to explain the above terminology. A locally convex space X is said subnormed, if it has a norm $\|\cdot\|$ majorizing its topology, i.e., for each $\alpha \in \mathfrak{a}$ there is a positive constant C_α such that

$$|x|_\alpha \leq C_\alpha\|x\| \quad \text{for all} \quad x \in X. \tag{4}$$

If X is not normed, then the norm $\|\cdot\|$ is necessarily discontinuous, and each metrizable subnormed space is normed. An example of non-normed complete subnormed space is the space $X = C[0,1]$ equipped with seminorms $(|\cdot|_\alpha)$, where α is a convergent sequence $\xi_i^{(\alpha)} \to \xi_0^{(\alpha)}$, $0 \leq \xi_i^{(\alpha)} \leq 1$, and $|x|_\alpha = \max_i |x(\xi_i^{(\alpha)})|$ (this family is non-saturated). In this example $\|\cdot\|$ is the usual supremum norm and all continuous endomorphisms of our space are also continuous with respect to $\|\cdot\|$, so that $L(X)$ is a normed space with respect to the operator norm induced by $\|\cdot\|$.

A bounded subset $B \subset X \in LC$ is said *maximal* if for any bounded subset $S \subset X$ there is a positive C_S such that $S \subset C_S B$ (a subset S is bounded if all continuous seminorms are bounded on it). A maximal bounded subset in the above example is the unit ball with respect to $\|\cdot\|$.

An LC-space X is called *bornologically normable* if there is a norm $\|\cdot\|$ on it such that all bounded sets coincide with bounded sets with respect to $\|\cdot\|$.

The locally convex spaces satisfying conditions of Theorem 2.1 are rather exceptional. An example of an LC-space which does not meet these conditions is the space $X = C[0,\infty)$ equipped with seminorms $|x|_n = \max_{0\leq t\leq n} |x(t)|$.

But even if $L(X)$ is LC-topologizable, there are no chances for the joint continuity of the map

$$(T, x) \mapsto Tx \tag{5}$$

from $L(X) \times X$ onto X. We have the following result ([9], Theorem 10.2).

Theorem 2.2. *Let X be a locally convex space, and suppose that $L(X)$ has a locally convex vector space topology such that the map* (5) *is jointly continuous. Then the space X is normed.*

Theorems 2.1 and 2.2 show that there is no reasonable locally convex topology on $L(X)$ if X is non-normed (in some instances it can be used the strong operator topology on $L(X)$, but then the map (5) is only separately continuous). Thus it makes a sense to consider only some subalgebras of $L(X)$ for which such a reasonable topology exists, instead of the whole algebra.

The following definition was given in [14, Definition 1].

Definition. Let X be a locally convex space. An *operator algebra* on X is a unital subalgebra A of $L(X)$ which is a locally convex topological algebra under some topology, and the map (5) is jointly continuous.

For such an operator algebra with topology given by a family $(\|\cdot\|_\mu)$ the condition (5) can be written as follows. For each $\alpha \in \mathfrak{a}$ there is a $\beta(\alpha)$ and a μ such that for all T in A and all x in X we have

$$|Tx|_\alpha \leq \|T\|_\mu |x|_{\beta(\alpha)}. \tag{6}$$

Formally speaking, there should be a constant factor on the right-hand of (6), but we can always renorm A so that this factor equals 1.

In the above definition we do not assume that neither X nor A is complete. However, we can take both completions and all operators in A extend to the completion of X with preserved relations (3) and (6). Thus, if necessary, we can assume, without loss of generality, that both X and A are complete.

The following definition was also introduced in [14].

Definition. Let X be as above. Call an operator T in $L(X)$ *topologizable*, if it belongs to some operator algebra on X.

Theorem 2.3 ([14]**, Theorem 5).** *Let X be a locally convex space. An operator T in $L(X)$ is topologizable if and only if for each $\alpha \in \mathfrak{a}$ there is a $\beta(\alpha) \in \mathfrak{a}$ and a sequence $(M_n^{(\alpha)})$ of positive numbers, such that*

$$|T^n x|_\alpha \leq M_n^{(\alpha)} |x|_{\beta(\alpha)} \tag{7}$$

for all x in X and all non-negative integers n (here by T^0 we understand the identity operator I on X). That means that all powers T^n are similarly continuous (a concept weaker than equicontinuity – it will be defined later on).

Example. Let X be the mentioned above space $C[0,\infty)$ and let φ be an arbitrary element in X. Then the operator $T_\varphi x = \varphi x$ is topologizable, and the shift operator T, given by $(Tx)(t) = x(t+1)$ is not topologizable.

It is shown in [14] that the sum and product of two commuting topologizable operators is topologizable, but it can happen that if T_1 and T_2 are topologizable, but non-commuting, then both T_1+T_2 and T_1T_2 are not topologizable. It can even happen that all operators in $L(X)$ are topologizable, while X is non-normed and so $L(X)$ is not an operator algebra. Such a space X was constructed by Bonet in [1].

In what follows we shall be concerned with joint topologizability of operators in $L(X)$, but first we need the following (already announced) concept.

Definition. Let S be a non-void subset of $L(X)$. We say that the operators in S are *similarly continuous*, if for each $\alpha \in \mathfrak{a}$ there is a $\beta(\alpha) \in \mathfrak{a}$ such that for all T in S and all x in X we have

$$|Tx|_\alpha \leq C_T^\alpha |x|_{\beta(\alpha)} \tag{8}$$

for some positive constants $C_T^{(\alpha)}$. Without loss of generality we can assume here that $\alpha \leq \beta(\alpha)$ (we write $\alpha \leq \beta$ if $|\cdot|_\alpha$ is continuous with respect to $|\cdot|_\beta$, i.e., there is a positive constant C such that $|x|_\alpha \leq |x|_\beta$ for all x in X).

One can easily see that the above concept does not depend upon a particular saturated system $(|\cdot|_\alpha)_{\alpha\in\mathfrak{a}}$ of seminorms giving the topology of X. Thus each function $\alpha \mapsto \beta(\alpha)$ from $\mathfrak{a}$ into itself, defines a subset $L_\beta(X)$ of $L(X)$ of all operators satisfying relation (8), so that this subset consists of similarly continuous operators. Since, by the formula (6), each operator algebra consists of similarly continuous operators, every such an algebra is contained in some $L_\beta(X)$. Also every operator in $L(X)$ belongs to some $L_\beta(X)$. It is convenient to consider only the functions $\beta(\alpha)$ satisfying $\alpha \leq \beta(\alpha)$. Such a function will be called admissible. Under this assumption (kept in the sequel) the identity operator belongs to all spaces $L_\beta(X)$. We shall introduce now a (natural) topology on these spaces. For each α in $\mathfrak{a}$ and each T in $L_\beta(X)$ we put

$$\|T\|_\alpha = \sup\{|Tx|_\alpha : |x|_{\beta(\alpha)} \leq 1\}. \tag{9}$$

The formula (8) shows that the right-hand expression in (9) is well defined. and we can easily see that it is a seminorm on $L_\beta(X)$. Formula (9) implies

$$|Tx|_\alpha \leq \|T\|_\alpha |x|_{\beta(\alpha)} \tag{10}$$

for all x in X, all α in $\mathfrak{a}$ and all T in $L_\beta(X)$. Note that if T belongs to different spaces of the form $L_\beta(X)$, the value of $\|T\|_\alpha$ depends not only upon T and α but also upon the chosen function $\beta(\alpha)$. In the case when it will be necessary we shall be denoting $\|T\|_\alpha$ also by $\|T\|_\alpha^\beta$ in order to indicate the used function $\beta(\alpha)$. Observe also that if Y is a vector subspace of $L(X)$, equipped with a family of seminorms $(||T||_\mu)$, so that the map (5) of $Y \times X$ into X is jointly continuous, then it can be continuously imbedded into some $L_\beta(X)$ for some admissible β. It can be easily seen by the use of formulas (6), (8) and (9). In particular each operator algebra can be continuously imbedded into some $L_\beta(X)$.

Proposition 2.4. *Assume that X is complete. Then $L_\beta(X)$ is a complete locally convex space under the seminorms* (9).

Proof. Suppose that (T_μ) is a Cauchy net of elements of $L_\beta(X)$. Let $x \in X$ with $|x|_{\beta(\alpha)} = 1$. The formula (10) implies that $T_\mu x$ is a Cauchy net in X. Consequently $T_\mu(x)$ is a Cauchy net for all x in X (if $|x|_{\beta(\alpha)} \neq 0$, then $T_\mu(x/|x|_{\beta(\alpha)})$ is Cauchy, if $|x|_{\beta(\alpha)} = 0$, then, by (10), $T_\mu x = 0$ and is also Cauchy). By the completeness of X there exists the limit

$$T_0(x) = \lim_\mu T_\mu x.$$

Clearly T_0 is additive and homogeneous on X. There exists also the limit $\|T_0\|_\alpha = \lim_\mu \|T_\mu\|_\alpha$ and the formula (10) implies that this formula is also satisfied by T_0 for all α in $\mathfrak{a}$. Thus T_0 is continuous and belongs to $L_\beta(X)$. The conclusion follows. □

We omit an easy proof of the following

Proposition 2.5. *For any two admissible functions β_1 and β_2 we have*

$$L_{\beta_1}(X) \cdot L_{\beta_2}(X) \subset L_{\beta_2 \circ \beta_1(\alpha)} \tag{11}$$

and for $T_i \in L_{\beta_i(\alpha)}(X), i = 1, 2,$ we have

$$|T_1T_2x|_\alpha \leq \|T_1\||T_2x|_{\beta_1(\alpha)} \leq \|T_1\|_\alpha\|T_2\|_{\beta_1(\alpha)}|x|_{\beta_2(\beta_1(x))}$$

so that

$$\|T_1T_2\|_\alpha \leq \|T_1\|_\alpha\|T_2\|_{\beta_1(\alpha)}. \tag{12}$$

Definition. An n-tuple $(T_1, \ldots, T_n) \subset L(X)$ is called *jointly topologizable* if there is an operator algebra $A \subset L(X)$ containing operators $T_1, \ldots, T_n$ (similarly we define a *jointly topologizable family* $S \subset L(X)$).

Proposition 2.6. *A finite set of mutually commuting topologizable operators is jointly topologizable.*

Proof. Let $(T_1, \ldots, T_n)$ be the set in question. There are admissible functions $\beta_1, \ldots, \beta_n$ such that all integral powers of T_i are in L_{β_i}. Denote by A the algebra of all polynomials in $T_1, \ldots, T_n$. By the formula (11) and an easy induction we obtain for all non-negative integers $k_1, \ldots, k_n$

$$T_1^{k_1} \cdot \cdots \cdot T_n^{k_n} \in L_\beta(X),$$

where $\beta(\alpha) = (\beta_n \circ \cdots \circ \beta_1)(\alpha)$. Since $L_\beta(X)$ is a linear space, we have $A \subset L_\beta(X)$. Provide A with the maximal locally convex topology, so that all seminorms $\|T\|_\alpha^{\beta_i}, i = 1, 2, \ldots, n, \alpha \in \mathfrak{a}$, are continuous.

The formulas (10) imply now that the relations (5) hold true and A is an operator algebra containing the operators T_i. Since A is finitely generated, by a result in [10], it is a topological algebra (the multiplication is jointly continuous). The conclusion follows. □

As mentioned earlier, the product of two non-commuting topologizable operators is not necessarily topologizable, so that the above result is not true for an arbitrary finite set of topologizable operators. It is also not true in general that an infinite set of mutually commuting topologizable operators is jointly topologizable as shown in the following example.

Example. Let $X = C[0,\infty)$ with seminorms $|x|_k = \max_{0\le t\le k}|x(t)|$. Define

$$[T_i(x)](t) = \begin{cases} x(i), & \text{if } 0 \le t \le i; \\ x(t), & \text{if } i \le t < \infty, \end{cases}$$

$i = 1, 2, \ldots$. The operators T_i belong to $L(X)$ and mutually commute. In fact, $T_iT_j = T_{\max(i,j)}$, so that they are also idempotents. However, there is no admissible function β such that all operators T_i are in $L_\beta(X)$. Otherwise, we have

$$|T_jx|_1 \le \|T_j\|_1 |x|_{\beta(1)} \tag{13}$$

for all x and j. Choosing $j > \beta(1)$ and x so that $x(t) = 0$ for $0 \le t \le \beta(1)$ and $x(j) = 1$ we have the right-hand of (13) equal to 1, while the left-hand equals to 0. The contradiction proves our assertion.

Now we shall give a necessary and sufficient condition in order that a countable set $S = \{T_1, T_2, \ldots\}$ of not necessarily commuting operators belongs to some operator algebra, i.e., it is jointly topologizable.

Theorem 2.7. *Suppose that $S \subset L(X)$ is countable. Then S is jointly topologizable if and only if there is an admissible function β such that all finite products of elements of S are in $L_\beta(X)$.*

Proof. The condition is necessary since each operator algebra is contained in some $L_\beta(X)$. On the other hand, denoting by A the smallest subalgebra of $L(X)$ containing S, we have $A \subset L_\beta(X)$. Thus it is sufficient to provide A with an algebra topology (i.e., with jointly continuous multiplication), which is stronger than the topology of $L_\beta(X)$ in order to have joint continuity of the map (5) from $A \times X$ to X. As this stronger topology we can take the maximal locally convex topology τ_{max}^{LC} on A because, by a result in [10], it gives joint continuity of multiplication on any countably generated algebra. □

Remark. If we have an algebra A contained in some $L_\beta(X)$, we do not know whether it has a stronger topology than that of $L_\beta(X)$ making it an LC-topological algebra. We even do not know whether it is topologizable with a jointly continuous multiplication.

So we ask the following

Problem. Suppose that for some admissible function $\beta(\alpha)$ there is an algebra A contained in $L_\beta(X)$. Does it follow that A is an operator algebra?

Observe that if for an algebra A of operators we have $A \subset L_\beta(X)$, then by Theorem 2.7 all operators in A are topologizable.

Having the spaces $L_\beta(X)$ we can easily see the proof of Theorem 2.2: if there is a locally convex topology on $L(X)$ such that the map (5) is jointly continuous, then there is an admissible function β such that $L(X) \subset L_\beta(X)(= L(X))$. It means, in particular, that for a fixed α_0 in $\mathfrak{a}$ we have

$$|Tx|_{\alpha_0} \le C_T|x|_{\beta_{\alpha_0}} \tag{14}$$

for all T in $L(X)$ and all x in X. Thus for each continuous linear functional f on X and a fixed $x_0 \in X$ with $|x_0|_{\alpha_0} = 1$, substituting in (14) $Tx = f(x)x_0$ we obtain

$$|f(x)| = |f(x)x_0| \le C(f, x_0)|x|_{\beta(\alpha_0)},$$

which mean that all continuous functionals on X are continuous with respect to the seminorm $|\cdot|_{\alpha_0}$. A result of Mazur and Orlicz ([7], see 2.22 on p. 139) implies now that X is a normed space and the conclusion of Theorem 2.2 holds.

3. Invariant subspaces and the Fell and Doran problem

Both topics shall be considered on the space $X = (s)$ of all (real or complex) sequences provided with the topology of coordinatewise convergence (it is Banach's notation; the German school denotes this space by ω). If $x \in X$, $x = (\alpha_i(x))_1^\infty$, then the topology in question can be given by the sequence of seminorms $|x|_k = \sum_1^k |\alpha_i(x)|$, $1 = 1, 2, \ldots$.

Körber ([4]) has shown that each operator in $L(X)$ has a proper closed invariant subspace. In [13] we have obtained the following result

Theorem 3.1. *Let X be the complex space (s). Then each operator in $L(X)$ has a proper closed invariant subspace which is either of dimension or of codimension 1. If X is the real space (s) then there are operators which do not satisfy above, but then they have proper closed invariant subspaces of codimension 2.*

Using this result we shall prove the following (new) result. Recall that a proper subspace of X is called hyperinvariant for an operator T in $L(X)$, if it is invariant for every operator commuting with T. An operator T in $L(X)$ is said non-constant, if it is not of the form $T = \lambda I$, where I is the identity operator and λ is a (real or complex) scalar. In the proof we shall make use of the fact that the space $X = (s)$ is reflexive (see [5], p. 283) and so every operator in $L(X')$ is of the form $T^*, T \in L(X)$ $((T^*f)x = f(Tx), x \in X, f \in X')$. Thus if Y is a proper closed subspace of X' invariant for an operator $T^* \in L(X')$, then $Y^\perp \subset X$ is a proper closed invariant subspace for T $(Y^\perp = \{x \in X : f(x) = 0 \quad \text{for all} \quad f \in Y\})$.

Theorem 3.2. *Each non-constant operator in $L(X)$ (X – a complex space (s)) has a proper closed hyperinvariant subspace.*

Proof. Let $T \in L(X)$. If T has a one-dimensional invariant subspace $\mathbf{C}x_0, 0 \neq x_0 \in X$, then $Tx_0 = \lambda x_0$ for some complex λ and $x_0 \in \ker(T - \lambda I)$. Put $X_0 = \ker(T - \lambda I)$, it is a non-zero closed subspace of X since $x_0 \in X_0$, and $X_0 \neq X$, since T is non-constant. For an arbitrary operator S in $L(X)$ satisfying $ST = TS$

and for an arbitrary element x_0 of X_0 we have $(T - \lambda I)Sx_0 = S(T - \lambda I)x_0 = 0$. Thus $Sx_0 \in X_0$, so that X_0 is invariant for S. Since S is an arbitrary operator in $L(X)$ commuting with T, X_0 is hyperinvariant for T.

If T has no one-dimensional closed invariant subspace, then, by Theorem 2.1 it has such a subspace X_0 of codimension one. Thus X_0 is a kernel of some (non-zero) continuous linear functional f_0 on X. It follows that if $f_0(x) = 0$ implies $f_0(Tx) = 0$, i.e., $(T^* f_0)(x) = 0$ (the conjugate operator on the dual space X'). Thus we have either $T^* f_0 = 0$, or f_0 and $T^* f_0$ have the same kernel and so are proportional. In both cases there is a complex λ such that $Tf_0 = \lambda f_0$. Consequently T^* has a one-dimensional invariant subspace in X' and, similarly as in the first part of this proof, its kernel is a proper closed subspace of X'. Let now S be an arbitrary operator in $L(X)$, commuting with T. Thus S^* commutes with T^* and, similarly as above, the kernel $Y \subset X'$ of T^* is invariant for S^*. Consequently, $Y^\perp$ is a proper closed subspace of X which is invariant for S. Since S was an arbitrary operator in $L(X)$ which commutes with T, $Y^\perp$ is hyperinvariant for T. The conclusion follows. □

Remark. The above result fails to be true for the real space (s). To see it we take the space X which is the complex space (s) considered as a real space. It is clear that X is topologically isomorphic to the real space (s). The set $L_{\mathbf{C}}(X)$ of all complex-linear continuous operators on X is a subalgebra of $L(X)$. Consider the operator $T_0 \in L_{\mathbf{C}}(X)$ given by $T_0 x = ix$, it is a non-constant operator in $L(X)$ commuting with all operators in $L_{\mathbf{C}}(X)$. The operator T_0 has no (closed or not) proper hyperinvariant subspace in X since the algebra $L_{\mathbf{C}}(X)$ is transitive on X, i.e., for every $x, y \in X, x \neq 0$, there is an operator T in $L_{\mathbf{C}}(X)$ with $Tx = y$.

Let X be a locally convex space and A an algebra over the same field of scalars ($\mathbf{R}$ or $\mathbf{C}$). A representation of A on X is a homomorphism h of A into $L(X)$. We assume that h is unital (i.e., $h(e) = I$) if A is has a unity e. A representation h is said irreducible if there is no closed proper invariant subspace for all operators in $h(A)$. Thus, setting $\mathcal{A} = h(A) \subset L(X)$, h is irreducible if the orbit

$$\mathcal{O}(h; x_0) = \{Tx_0 : T \in \mathcal{A}\}$$

is a dense subspace of X for all non-zero elements x_0 in X (otherwise the closure of such an orbit is a closed invariant subspace for all operators T in $\mathcal{A}$).

A representation h is said k-fold irreducible, if for each k-tuple $(x_1, \ldots, x_k)$ of linearly independent elements of X, the orbit

$$\mathcal{O}(h; x_1, \ldots, x_k) = \{(Tx_1, \ldots, Tx_k) \in X^k : T \in \mathcal{A}\}$$

is dense in X^n.

Finally, call h totally irreducible, if it is k-fold irreducible for all natural k. In other words, h is totally irreducible if the algebra $\mathcal{A}$ is dense in $L(X)$ in the strong operator topology.

Problem (Fell and Doran [2], Problem II, p. 329). Let h be an irreducible representation of an algebra A on a locally convex space X, such that the commutant of the family of operators $h(A)$ consists only of scalar multiples of the identity operator. Does it follow that h is totally irreducible?

The condition concerning triviality of the commutant of $h(A)$ is necessary here, for if X is a Banach space and $T \in L(X)$ has no proper closed invariant subspace, then the representation $p \mapsto p(T)$ of the algebra P of all polynomials on X is irreducible, but not totally irreducible, since the commutative algebra $\{p(T) \in L(X) : p \in P\}$ cannot be strongly dense in the non-commutative algebra $L(X)$.

Theorem 3.3 ([11]). *The Fell and Doran problem has a positive solution on the real or complex space* (s).

Note that up to now the space (s) is the only known infinite-dimensional locally convex space for which the Fell and Doran problem has a positive answer. No positive or negative answer to this problem is known for any infinite-dimensional Banach space (the general belief is that there should exist a counterexample, perhaps on every infinite-dimensional Banach space). It is also the only infinite-dimensional separable locally convex space for which all endomorphisms are known to have proper closed invariant (and hyperinvariant) subspaces.

The purely algebraic counterparts of Theorems 3.1–3.3 are known for complex spaces and algebras. Schaefer [8] has shown that each linear map of an infinite-dimensional complex vector space has a proper invariant subspace, the present author recently has shown (the paper is submitted) that such a non-constant map has a proper hyperinvariant subspace if and only if its spectrum is non-void. The algebraic version of Theorem 3.3 is contained in the famous Jacobson's density theorem (see [3], pp. 283–286).

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